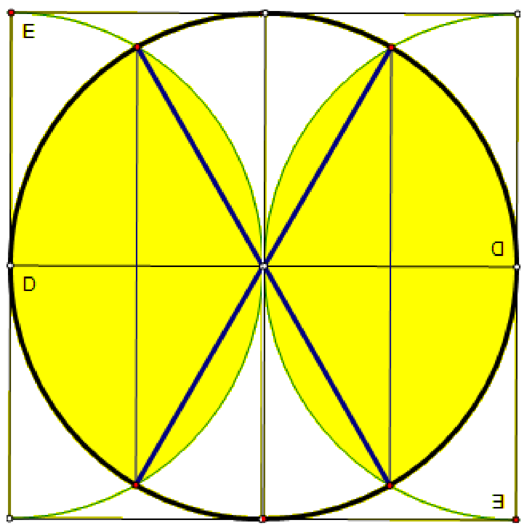
MATH 7
PRE-ALGEBRA &
PROBLEM-SOLVING



BOOK I

LAKESIDE MIDDLE SCHOOL
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2009-2010

MATH 7
PRE-ALGEBRA &
PROBLEM-SOLVING

BOOK I

Lakeside Middle School, Seattle, WA

Introduction

This math book is designed to reinforce the mathematics you already know and to introduce you to algebra and some other new topics. Throughout the book we aim to provide you with enjoyable and useful practice in problem-solving and to guide you in the exciting study of the tools of algebra. The experience we intend students to have is one of strengthening their existing understanding, extending their knowledge, and expanding their problem-solving powers while being challenged and having some fun.

The format of the book has the following features:

- Brief explanation and introduction prior to most problems sets.
- Introduction of new vocabulary in bold text in the context in which it will be used.
- Example problems with solutions for most problem sets.
- Over a hundred assignments which contain all or most of the following features:
    - A group of problems which practice the <u>current</u> topic.
    - A group of problems which practice concepts in <u>previous</u> problem sets.
    - Several seemingly random problems which provide a <u>stretch</u> from the current or previous (or even future) topics. These problems will be marked with the * symbol.

We advise students to adopt the following habits as a way to make the best use of their experience with this material.

- Do all the problems with care, even if you think you already understand the material. Repeated practice will increase your comfort, skill, and accuracy.
- Understand the concepts behind the results, rather than simply memorizing the results.
- Keep an ongoing, legible notebook or binder section with notes on concepts and techniques you are learning and examples of problems.
- When you miss problems on your first attempt, go back and re-do them. Nothing strengthens your grasp of difficult material and builds your confidence more than recovering from your own mistakes.
- Become articulate and understandable at explaining mathematics, both out loud and on paper. Learn the vocabulary and notation so that you can explain your ideas with precision and even eloquence, while still being comprehensible to your friends, your family, and your teacher. Learning to express mathematics clearly will help you understand it better and help you avoid the mistakes that arise from haste and carelessness. (That is why teachers are always urging you to "show your work.")

Good luck, and have fun!

Acknowledgements

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Chapter 1: Combinatorics

Mathematics is a vast landscape of adventure, discovery, and fun. The language we use to talk about it has an enormous amount of vocabulary, words we use to describe our experience and the underlying concepts. In this book, we will define or explain new words in bold type at the beginning of assignments.

Assignment #1.1: Introduction; Multiplication Principle

To find the total number of choices for an event, we multiply the number of choices for each part of the event. For example, if there are 2 ways to get from Seattle to Auburn, and 5 ways to get from Auburn to Tacoma, then there are $2 \cdot 5 = 10$ different ways to get from Seattle to Tacoma via Auburn.

Example: The bookstore sells 3 kinds of notebooks and 5 different kinds of mechanical pencils. If Abe wants to buy a notebook and a pencil, how many different ways can he do this?
Solution: First, let us choose a notebook. There are 3 different ways to do this. To complete the pair, we can choose any one of 5 pencils. Thus there are $3 \cdot 5 = 15$ different possible pairings.

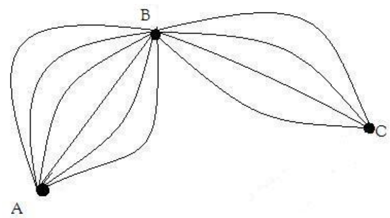
Example: The bookstore sells 3 kinds of notebooks, 5 different kinds of mechanical pencils, and 7 different colors of pens. If Olivia wants to buy one item of each kind, how many different ways can she do this?
Solution: If we start with each of the 15 different pairs in the last example, there are 7 ways to complete it by adding a pen. Thus, there are $15 \cdot 7 = 3 \cdot 5 \cdot 7 = 105$ different collections of school supplies Olivia could buy.

Example: The bookstore sells 3 kinds of notebooks, 5 different kinds of mechanical pencils, and 7 different colors of pens. If Atty wants to buy two different items, how many different ways can she do this?
Solution: There are three possible cases. Either Atty buys a notebook and a pencil, or she buys a notebook and a pen, or she buys a pencil and a pen. There are (3)(5) different ways to buy a notebook and pencil, (3)(7) different ways to buy a notebook and pen, and (5)(7) different ways to buy a pencil and a pen. Thus there are $15 + 21 + 35 = 71$ different ways for Atty to make a purchase.

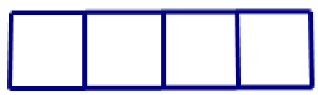
1.1 Exercises:

1. Ellie has four sweaters and three pairs of jeans. How many outfits can she make from these seven items?
2. If we toss a coin three times, how many different sequences of heads and tails are possible?

3. There are three towns in Gaussland: Adamsville, Beaverton, and Centerview. Six roads go from Adamsville to Beaverton and four roads go from Beaverton to Centerview. (See figure below.) How many ways are there to get from Adamsville to Centerview?



4. Each square in the grid (below) is colored either red or white. How many different ways are there to do this? (Note: The grid cannot be moved or rotated.)



5. The combination for a lock consists of three digits 0-9. How many different combinations are possible if the digits may repeat?

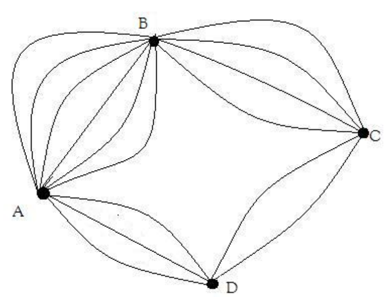
6. The combination for a lock consists of three digits 0-9. In how many ways can a combination be made if the digits may repeat, but if each digit must be odd?

7. The combination for a lock consists of three digits 0-9. How many different combinations are possible if no digit can be used more than once?

8. Every day Jack P. eats a sandwich and a piece of fruit for lunch. The sandwich is always cheese, tuna, or salami. The fruit is an orange or an apple. How many different lunches can Jack P. have?

9. A deli specializes in gourmet sandwiches. Each sandwich has two slices of the same kind of bread, one or two kinds of meat and possibly a condiment. The choices for bread are white, wheat, rye, or poppy seed. The choices for meat are ham, turkey, or salami. Finally, the sandwich can have mayonnaise, mustard, or no condiment. How many different sandwiches can this deli make?

10. A new town, Deerboro, and several new roads are built in Gaussland. (See below.) Now how many ways are there to get from Adamsville to Centerview?



11. The Gaussian alphabet consists of the three letters X, Y, and Z. A word in this language is any sequence of no more than four letters. (Also, every such sequence is a word. For example, XX is a word, Y is a word, ZZYX, etc.) How many different words does the Gaussian language contain?

12. Every day Jason eats a sandwich (cheese, tuna, or salami) and either one or two pieces of fruit chosen from an assortment of apples, oranges and bananas. If he eats two pieces of fruit, they may be the same or different. How many different lunches can Jason make?

13. *Several bacteria are placed in a jar. One second later, each bacterium divides in two. The next second each of the resulting bacterium divides in two, and so forth. After one minute, the jar is full. When was the jar only half full?

Assignment #1.2: The Pigeon Hole Principle

Another useful counting technique is the **Pigeon Hole Principle**. This is the following: If one must put N + 1 pigeons into N pigeon holes, then some pigeon hole must contain two or more pigeons.

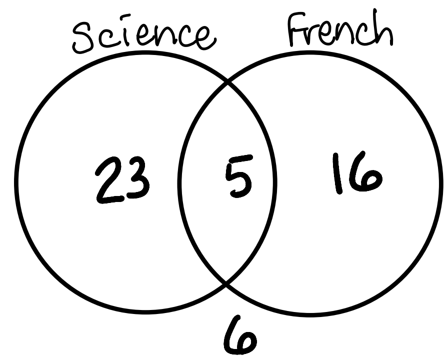
<u>Example</u>: A bag contains marbles of two colors, maroon and gold. What is the smallest number of marbles which must be drawn from the bag so that among these marbles there are two of the same color?
<u>Solution</u>: Three marbles will suffice. The first two marbles one draws out could be different colors, but the third must match one of the first two. (This may not seem to have much to do with pigeons, but it does. Think of the marbles as the pigeons and of the colors as the pigeon holes.)

<u>Example</u>: A million pine trees grow in the forest. It is known that no pine tree contains more than 600,000 pine needles. Show that two trees must have the same number of needles.
<u>Solution</u>: We have 600,001 different "pigeon holes", the different possibilities for numbers of pine needles on a tree (everything from 0 needles to the full 600,000), and one million trees. Since there are more trees than there are possibilities for differing numbers of needles, by the Pigeon Hole Principle there must be at least two trees with the same number of needles.

Still another useful counting technique is to draw a Venn diagram.

Example: There are 50 students in a school. Of these, 28 take science, 21 take French, and 5 students take both. How many take neither?

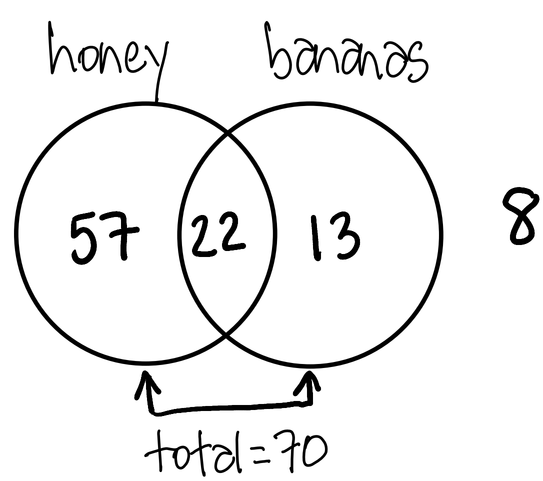
Solution:
Five students are in the overlapping area since they take both science and French. That leaves 28 – 5 or 23 who take only science and 21 – 5 or 16 who take only French. Adding 23, 5 and 16 totals 44 students so there must be 6 students out of the 50 taking neither class.



Example: Of 100 kindergarteners surveyed at Sidelake Elementary, 8 don't eat peanut butter (PB) sandwiches, 57 prefer only honey on their peanut butter sandwiches, and 22 prefer honey and bananas with their peanut butter. How many of these 100 students like only bananas on their peanut butter sandwiches? How many like honey with their PB?

Solution:
Eight students are not in the circles since they do not eat PB sandwiches. Twenty-two are in the overlap since they like both honey and bananas on their sandwiches. This leaves 70 students to be distributed into the two circles (100 – 8 – 22 = 70). We know that 57 prefer just honey so that fills the honey circle. This leaves 13 (70 – 57 = 13) students who prefer just bananas with their peanut butter.

To figure out how many students like honey with their PB we must include the students who like both bananas and honey as well as the students who like just honey. From the diagram we see that this is 57 + 22 or 79 students. Alternatively, 100 – 8 – 13 = 79.



1.2 Exercises:

1. A drawer contains ten socks with one pair of each of the following colors: brown, blue, black, tan, and white. If Sofia reaches into a drawer in which all these socks are jumbled together, what is the least number of socks she must remove to guarantee that she has a matching pair?

2. Of the 80 7^{th} grade students at Lakeside, 43 play only soccer in the fall, 5 play soccer and run cross-country, and 14 are not involved in either school sport. How many students are on the cross country team?

3. If the digits can be used more than once, how many positive even three digit integers can be created using the digits 2, 3, 4, 5, 7, and 9?

4. Seattle has two Lake Washington bridges, 520 and I-90. A survey of 1,000 people who commute across Lake Washington showed that 350 use only the 520 bridge and 475 use only the I-90 bridge. How many commuters use both bridges in their commute?

5. The integers 0-9 are on a circular lock (like the one pictured) so that 0 and 9 are next to one another. How many three digit combinations are possible if no two consecutive numbers in the combination are allowed to be the same or adjacent on the lock?

6. The city of Pascalidonia has five million inhabitants. If it is known that no person has more than one million hairs on his or her head, explain how you know that two residents of the city must have the same number of hairs on their heads.

7. How many different two-letter arrangements are possible using the 26 letters of the English alphabet? Assume you can use the same letter twice.

8. Morgan is taking a T/F test and is guessing at every answer. If there are twelve questions on the test, how many different sequences of answers are possible?

9. How many whole numbers from 100 – 600 inclusive contain the digit 5 exactly once?

10. How many different license plates are possible with 2 letters followed by 4 digits if no letter or digit is used twice?

11. *In a deep tunnel beneath the city of Rome, Matthew found the following symbols on a wall: (II, III) $\Rightarrow$ VI (X, III) $\Rightarrow$ XXX (VII, X) $\Rightarrow$ LXX

The fourth set of symbols was incomplete: (IV, IV) $\Rightarrow$

What are the missing characters?

Assignment #1.3: Permutations

A **permutation** is an arrangement of items created by reordering (or possibly removing) them.
Example: If Angel, Ben, and Carly run a race, how many different possible orders are there for their finishing times?
Solution: One solution is to list all the possible arrangements: ABC, ACB, BAC, BCA, CAB, and CBA. Thus, there are 6 orders in which Angel, Ben and Carly could finish. Of course, in a race with more than three people, this method of solution could become a bit tedious. Instead, we notice that there are 3 possible choices for the winner of the race. Once that person is known, there are 2 possibilities for the second place finisher, and after that, the remaining person must come in last. Hence, there are (3)(2)(1) = 6 possible outcomes for the race.

Example: If Angel, Ben, Carly and Daniel run a race, how many different possible orders are there for their finishing times?
Solution: This time there are (4)(3)(2)(1) = 24 possible outcomes.

In each of the examples above, we were counting permutations. In order to speak of permutations, it is useful to have some additional vocabulary. If n is a natural number, the number **n!** (pronounced **n factorial**) is the product $n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$. For convenience, we define $0! = 1$.

Notice that the answer to the examples above were $3! = 6$ and $4! = 24$, respectively. In general, for n distinct objects, there are n! permutations.

More formally, we can say that a **permutation** is an ordered list without repetitions, perhaps missing some elements. This emphasizes the possibility that we allow some elements to be removed. This possibility will be specified when applicable.

Example: If five people run a race, in how many ways could 1^{st}, 2^{nd}, and 3^{rd} place be awarded?
Solution: Here we are asking for the number of 3 person permutations out of 5 elements. Since there are 5 candidates for the first position, 4 candidates for the second position, and 3 candidates for the third position, we can say that the number of three-person permutations is (5)(4)(3) = 60.

1.3 Exercises:

Evaluate:

1. $6!$ **2.** $\frac{7!}{6!}$ **3.** $\frac{101!}{99!}$ **4.** $\frac{8!}{5!3!}$

5. List all of the three-letter permutations possible using the letters of the word "MAT".

6. List all of the three-letter permutations possible using the letters of the word "MOM".

7. How many four-letter permutations are possible using the letters of the word "MATH"? Write out at least six of these permutations.

8. List all of the four-letter permutations possible using the letters of the word "NOON".

9. In a survey of 100 people, 31 own a dog, 6 own both a dog and a cat, and 40 own neither a dog nor a cat. How many of the people surveyed own a cat?

10. How many different ways can one arrange the Jack of Hearts, Queen of Hearts, King of Hearts, and Ace of Hearts?

11. How many different ways can Lindsey arrange her math book, English book, science book, history book, art portfolio, and foreign language book on her shelf?

12. How many six-letter permutations can be made from the word "HAMLET"?

13. A department store want to have all possible two and three letter monograms ready on the towels its sells. A two letter monogram consists of a first name initial and a last name initial, while a three-letter monogram also includes a middle name initial. How many different monograms would this require?

14. Bill, the dashing pirate king, needs three matching earrings to complete his outfit. He reaches for a bag filled with diamond studs, gold hoops, silver hoops, and large dangling rubies. How many must he draw out in order to ensure having three earrings that match?

15. Six students and their two adult advisors are arranged in a line shoulder to shoulder for a group photo. In how many ways can they be arranged with an adult at each end?

16. A survey of the 80 7^{th} grade students at Lakeside found that all of the students help with dishes or garbage chores at home. Forty-six take out the garbage at home and 23 help with the dishes and garbage. How many students have only the dishwashing duty?

17. *Each of the marbles, A, B, and C is colored one of three colors. One of the marbles is colored white, one is colored red, and one is colored blue. Exactly <u>one</u> of these statements is true:

a) A is red.
b) B is not blue.
c) C is not red.

What color is marble B?

Assignment #1.4: More Permutations

In general, for n objects taken r at a time, there are n(n – 1)(n – 2)…(n – r + 1) permutations. This changes, however, if some of the items are indistinguishable.

Example: How many distinct ways are there to arrange the letters of the word "BANANA"?
Solution: We need the permutations of 6 things (i.e. 6!), divided by the number of ways to permute the three identical A's (i.e. 3!) and the two identical N's (i.e. 2!). The number of ways is $\frac{6!}{3!\cdot 2!}=\frac{720}{6\cdot 2}=\frac{720}{12}=60$. More efficiently, we could notice that the fraction can be reduced: $\frac{6!}{3!\cdot 2!}=\frac{6\cdot 5\cdot 4}{2}=6\cdot 5\cdot 2=60$.

1.4 Exercises:

Find *N*.

1. $\frac{8!}{5!}=N$

2. $\frac{6!}{4!\cdot 2!}=N$

3. $\frac{5!}{3!\cdot 2!}=N$

4. $4\cdot 3!=N!$

5. How many six-letter permutations can be made from the word “SUMMER”?

6. In how many ways can the letters of the word “TUMULT” be arranged?

7. How many six-letter permutations can be made from the word “HANNAH”?

8. In how many ways can the letters of the word “ALASKA” be arranged?

9. Ten sprinters run the 100 yard dash. How many different ways can they possibly finish first, second and third?

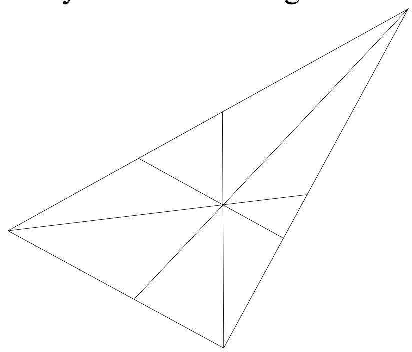
10. Destiny knows 7 songs. She wants to pick 3 for her upcoming performance. How many different permutations of 3 songs can she come up with for her set?

11. At Pigs-R-Us there are 50 guinea pigs of many different kinds for sale. Each guinea pig, however, has a curly coat, red eyes, or floppy ears. Two guinea pigs have all three of these features. Seven have curly coats and red eyes, eight have red eyes and floppy ears, and twelve have curly coats and floppy ears. If 20 guinea pigs have curly coats and 20 have red eyes, and if Mr. Chen wants a guinea pig with floppy ears, how many different animals does he have to choose from?

12. How many three digit numbers contain all three of the digits 4, 5, and 6?

13. How many different ways are there for Andy, Braeden, Christina, and Devin to seat themselves at a square card table? (Note: If Andy has Braeden on his right, Christina on his left, and Devin across from him, that is ONE arrangement no matter which seat he occupies.)

14. *How many triangles can you find in this figure?



Assignment #1.5: Combinations

If you have six people who want to play tennis, then it *seems* as if the number of pairings of two of these people to play singles tennis is

$$6 \cdot 5 = 30$$

But this method has a flaw; it is counting some pairings twice because it treats the person X playing person Y as different from person Y playing person X. In permutations, order matters and counts as a separate arrangement. In this case, however, we seek a count of arrangements in which the order of the objects does not matter, i.e. it does not produce a new arrangement. Such arrangements are called **combinations**.

Tennis is a good example. Person X playing person Y should not be counted as different from person Y playing person X. So to count the distinct combinations of two players taken from a pool of six players, we should divide by 2 (which is the number of permutations of 2 players). In this case, the number of all distinguishable pairings is 30/2 or 15.

Thus, in this example, if you have six persons for tennis, then the number of pairings of two of these people to play singles tennis is:

$$\frac{6 \cdot 5}{2} = 15\,.$$

Example: How many ways are there to choose a president, vice-president, and treasurer from a group of 10 club members?
Solution: There are (10)(9)(8) = 720 ways to choose the officers.

Example: How many ways are there to choose three class representatives from a group of 10 candidates?
Solution: There are $\frac{(10)(9)(8)}{3 \cdot 2} = \frac{(10)(3)(4)}{1} = 120$ ways to choose the representatives.

1.5 Exercises:

Find N.

1. $7! = N$

2. $3 \cdot 2 \cdot 1 = N!$

3. $5!\cdot 2! = N$

4. $7 \cdot 6! = N!$

5. $\dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7!} = N$

6. $\dfrac{11!}{9!} = N$

7. $\dfrac{9!}{3!\cdot 2!} = N$

8. $\dfrac{12!}{6!\cdot 4!} = N$

9. In how many ways can the letters of the word "ALABAMA" be arranged?

10. In how many ways can you choose 5 students out of 8 students where the order of the selection matters (i.e. a different order counts as a different arrangement)?

11. In how many ways can you choose 5 students out of 8 students where the order of the selection does not matter (i.e. a different order does not count as a different arrangement)?

12. An unnamed middle school has thirty-two 5th graders, sixty-four 6th graders, seventy-eight 7th graders and eighty 8th graders. How many different 4-person committees can be formed with one student from each of the grade levels?

13. In a survey of 100 Upper School foreign language students, two were found to take French, Chinese, and Spanish. Five take French and Spanish, seven take Chinese and Spanish, and three take Chinese and French. If 39 take Chinese and 42 take Spanish, how many study French?

14. Mr. Hopson asked his 16 science students to line up at the flight simulator. In how many different ways could his students line up?

15. How many combinations of triple scoop ice-cream cones are possible if you have 31 flavors from which to choose? (Assume that you cannot have more than one scoop of any flavor and that the order of the scoops does not matter.)

16. If I have three young pear trees and two new apple trees, how many ways are there to plant these in a row along my back fence?

17. A planning committee of four students is being organized for next year's talent show. Twelve students volunteer to serve. How many different groups of four could make up the committee from these volunteers?

18. How many different ways could a team of seven students be formed if the team must have 4 boys and 3 girls and the entire group is made of 5 boys and 7 girls?

19. *Molly Rose was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead she subtracted 9 and then divided the result by 3, obtaining an answer of 43. What would her answer have been if she had worked the problem correctly?

Assignment #1.6: Practice with Permutations and Combinations

1.6 Exercises:

1. Peter and three of his friends are at a restaurant. They intend to select four different menu items to share. In how many different ways can they do this if the menu includes 12 offerings?

2. Paul, the winner of a karaoke contest, gets to choose 5 CDs from a collection of 10 selections. How many different groups of 5 CDs could he choose?

3. In how many distinct ways can the letters of the word "REPLETE" be arranged?

4. There are 10 rides at the carnival. You are going to ride each of them once. In how many different ways could you do this?

5. There are 10 different rides at the carnival, but you only have enough money to ride five. How many possible sequences of five different rides could you choose?

6. There are 10 different rides at the carnival, but you only have enough money to ride three. How many possible sets of three rides could you choose? (Assume that you could go on the same ride all three times if you wanted to do that.)

7. Mr. Jamieson asks for 3 volunteers from his advisory to help with recycling. All 8 of Mr. Jamieson's advisees volunteer. To choose the volunteers, Mr. Jamieson puts all 8 names into a hat and randomly picks 3. How many possible combinations of volunteers are possible?

8. Four players from the roster of twelve have the honor of marching single file at the head of the New Year's parade. In how many different ways could this happen?

9. Glog, a five-footed Martian, gets dressed one morning in the dark. Choosing socks from a jumble of 6 brown ones, 4 red ones, 7 purple ones, and 13 yellow ones, he continues to draw socks out of the drawer until confident that he must have five of the same color. How many socks does Glog select to know he has the needed matching set?

10. A school decides it wants to make 3-letter identification codes for each student using only the standard vowels from the alphabet. Each letter may be used only once in a code. How many unique codes could be made using this method?

11. *Using only standard American coins, how many ways are there to make change for 27¢?

12. *Nathan has nine gold coins, all identical in appearance. Unfortunately, only eight are pure gold; the ninth contains some iron and is heavier than the others. If Nathan has only a balance scale, what is the least number of weighings he can use to identify the counterfeit coin?

Assignment #1.7: Permutations and Combinations – Using a Calculator

We have seen that if you have a collection of **n** distinguishable objects, then the number of permutations of **r** of them (r < n) is n(n – 1)(n – 2)… (n – r + 1). A formal statement of this is often made with subscript notation. Let us denote the number of permutations of n distinct objects taken r at a time with the expression ${}_nP_r$. Then

$$\begin{aligned} {}_nP_r &= n(n-1)(n-2)\ldots(n-r+1) \\ &= n(n-1)(n-2)\ldots(n-r+1)\cdot\frac{(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

That is, ${}_nP_r = \dfrac{n!}{(n-r)!}$.

Similarly, if you have a collection of **n** distinguishable objects, then the number of ways you can pick a number **r** of them (r < n), while eliminating the different permutations of the **r** objects, is given by the combination relationship:

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

1.7 Exercises:

Use your calculator to solve the following problems.

1. What is the value of $\dfrac{9!}{4!3!}$?
2. How many 10-letter "words" can be made from the word "FLAGELLATE"?
3. Ms. Hon has a closet containing 22 sleeping bags. She needs to choose 8 of these to lend to students in her advisory for outdoor trips. How many different sets of 8 sleeping bags could be picked from her closet?
4. If seventy different committees of the same size could be formed by choosing some given number of individuals from a group of 8 people, how many people would be on a committee? (Hint: order does not matter in the choosing of a committee.)
5. How many different teams of three volunteers for cooking can be chosen from a group of 10 hikers?
6. How many ways are there to choose a seventh grade teacher (from among 12) and a seventh grader (from among 80) to plan activities for the seventh grade?
7. How many ways are there to choose a faculty member and two seventh grade students from the same group as in problem 6?

8. Sadie, Austin A., Kailee and Ryan have tickets for four reserved seats in a row at Benaroya Hall. In how many different ways can they seat themselves if Sadie and Kailee must be seated next to each other?

9. Sadie, Austin A., Kailee, Ryan and Katherine have tickets for five reserved seats in a row at Benaroya Hall. In how many different ways can they seat themselves if Sadie and Kailee must be seated next to each other?

10. Sadie, Austin A., Kailee, Ryan and Katherine have tickets for five reserved seats in a row at Benaroya Hall. In how many different ways can they seat themselves if the three girls must be seated next to each other?

11. Sadie, Austin A., Kailee, Ryan and Katherine have tickets for five reserved seats in a row at Benaroya Hall. In how many different ways can they seat themselves if the three girls would like to sit together and the two boys would also like to sit together?

12. Radio station WAVE plays 3 songs and 4 commercials every 20 minutes. For one 20 minute segment, the disc jockey chooses three different songs to be played and the producer chooses four different commercials. The producer then decides that the order will be song, commercial, two songs, and three commercials. How many possible arrangements are there for this 20 minute segment?

13. How many different four-digit numbers can be made from the digits of 2009?

14. A car holds 6 people, but only two of those people can drive. How many ways can the people be seated in the car so that a driver is in the driver's seat?

15. *As a fundraising project, McLean wants to sell 50 identical pens with the Lakeside logo in groups of two or three. How many different ways are there to package the pens?

16. *Cross out 10 digits from the number 1234512345123451234512345 so that the remaining number is as large as possible.

Assignment #1.8: Still More Permutations and Combinations

1.8 Exercises:

1. How many different four-digit numbers can be obtained using any four of the digits 2, 3, 4, 4, and 4?

2. How many positive four digit integers contain each of the digits 1, 2, 3, and 4 exactly once?

3. Mary must read two books for her literature course. She must read any two of five biographies. How many different sets of two books could she choose?

4. Wallis must read five books for his literature class. He must read any one of three biographies, any two of four mysteries, and any two of five science fiction/fantasy books on his list. How many different sets of five books can he choose?

5. How many different four digit numbers greater than 6000 can be formed from the digits 0, 1, 3, 5, 6, and 8 if no digit may be used more than once?

6. A school organization consists of 5 teachers, 7 parents, and 6 students. A subcommittee is formed consisting of 2 teachers, 3 parents, and 3 students. How many different ways can this subcommittee be formed?

7. A compact disc player is programmed to play compact discs in random order. If it is loaded with 7 CDs, what is the number of different ways the compact discs can be played?

8. Snow White would like some help washing apples. She asks for three volunteers from among her friends, the seven dwarfs. How many different sets of three dwarfs might volunteer?

9. How many different 3 digit numbers can you make from the digits 6, 7, 8 and 9 if no digit may be used more than once in a number?

10. How many different 3 digit numbers can you make from the digits 6, 7, 8 and 9 if each number <u>must</u> include a repeated digit? (Yes, it is possible to use one digit more than twice in a number.)

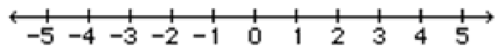
11. King Arthur, along with several of his knights (King Pellinore, Sir Lancelot, Sir Gawain, Sir Lionel, Sir Kay, Sir Galahad, and King Ban) are seated at the Round Table. How many different ways can they do so?

12. There are sixteen desks in the math classroom, but only ten students in one Math 7 class. In how many different ways could they seat themselves in the room?

Chapter 2: Integers; Expressions and Equations with Addition and Subtraction

The **integers** consist of the whole numbers (0, 1, 2, 3, 4, …) and the negatives of these numbers, i.e. (-1, -2, -3, -4, …).

The integers can be neatly pictured on a number line, as shown below.



This number line can be used to easily determine if a number is smaller or larger than another. To say that x is smaller than y is the same as saying that x is to the left of y on the number line.

To express the negative integers, we use the "-" sign. This sign is used for several ideas.

1) The "-" sign means "negative". Thus -2 means the number "negative two".
2) The "-" sign means "the opposite of". The opposite of 3 is -3. The opposite of -4 is 4. The opposite of 0 is 0. In general, the **opposite** of x is the number you would add to x to get zero. We designate this number as $-x$. So $x + -x$ is 0. (Another way of referring to the opposite of a number is to call it the **additive inverse** of the number.)
3) When used between two expressions, the "-" sign means <u>subtraction,</u> which we will define as "adding the opposite". So $6 - 8 = 6 + -8 = -2$.

This helps us with expressions like $2 - (-(-9))$. The innermost -9 means "negative 9". Then $(-(-9))$ is the additive inverse of -9, which is 9. Thus we have $2 - 9$, which we interpret as $2 + -9$, which is -7.

Note that the ideas in 1) and 2) above can be equivalent. -2 (negative two) is fortunately the same thing as -2 (the additive inverse of two). Again realize, however, that expressions like $-x$ are not necessarily negative. If $x = -7$, then $-x$ is positive.

The **integers** have certain properties when being added. You have known these properties all your life, but perhaps never gave them a formal name. In the following list, x, y, and z are any integers.

Commutative property — You can add in either direction; the result is the same.

$$x + y = y + x$$

Associative property — When adding several integers, you may group as you please.

$$(x + y) + z = x + (y + z)$$

Identity property

There exists an additive identity (0). This means that there is a number called 0 which can be added to any x to get x itself. (Hence the word *identity*.)

$$x + 0 = x$$

Inverse property

Every integer has an additive inverse, another integer which when added to the original integer gives a sum of 0, the additive identity.

$$x + (-x) = 0$$

Note: The commutative and associative properties do not apply to subtraction. The identity property does apply to subtraction.

$$x - 0 = x$$

Assignment #2.1: Evaluating Expressions

A **number** is an idea of quantity; a **numeral** is the way you symbolize it. For example, 15 and XV are both numerals to show the idea of "fifteen". We usually use Arabic numerals (like 15) for numbers whose values never change (i.e. **constants**), and letters of the alphabet for unknown numbers (which we call **variables**, since their values may vary).

Any collection of numerals is said to be a **numerical expression**, or simply an **expression**. For example, 5 – (6 – 2) is a numerical expression. When you replace a numerical expression with its simplest equivalent numeral, we say you have simplified or **evaluated** the expression.

When evaluating expressions with addition and subtraction, work from left to right. You should recall that when subtracting integers, we define subtraction to be adding the opposite.

Example: 6 – 8
Solution: 6 – 8
6 + -8
-2

Example: -6 – 8
Solution: -6 – 8
-6 + -8
-14

Example: -6 – (-8)
Solution: -6 – (-8)
-6 + -(-8)
-6 + 8
2

When there are parentheses in an expression, evaluate the parts in parentheses first.

Example: Evaluate 5 – (4 – 2)
Solution: 5 – (4 – 2)
5 – (2)
3

Example: Evaluate (-10 – 2) – [1 – (-4 – 6)]
Solution: (-10 – 2) – [1 – (-4 – 6)]
(-10 + -2) – [1 –(-4 + -6)]
(-12) – [1 – (-10)]
-12 – [1 + -(-10)]
-12 – [1 + 10]
-12 – [11]
-12 + -11
-23

2.1 Exercises:

Compute (without a calculator):

1. (-12 + 5) + (-19 + 12)

2. (-18 + -18) – (-7 + 9)

3. (-17 + 1) + 16

4. (10 + -13) – [(0 + -10) – 7]

5. -17 + (1 + 16)

6. [2 – (17 + -8)] – (-3 + 10)

7. (13 + 16) + -(1 + -2)

8. (4 – 19) – [(-3) – 5]

9. (14 – 7) + 7 – (-13)

10. 4 – 19 – (-3) – 5

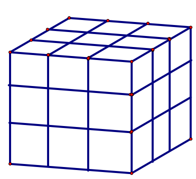
11. 1 – (-19) + (-13) – (-4)

12. 10 – (8 + -17) – 6

13. 0 – 1 – 2 – 3 – 4

14. –(5 – [(-2) + 17 – (-6)])

15. *A large cube is made up of 27 smaller cubes. If the large cube is painted red on all sides:



a) How many of the 27 small cubes would be painted on exactly 1 side?

b) How many of the 27 small cubes would be painted on exactly 2 sides?

c) How many of the 27 small cubes would be painted on exactly 3 sides?

d) How many of the 27 small cubes would be painted on exactly 4 sides?

Assignment #2.2: Expressions with Variables

An expression with variables is sometimes called an **algebraic** expression. To evaluate such expressions, we need to substitute the actual values of the variables. If x = 6, then x + 3 can be evaluated to 6 + 3, which is 9. When there are parentheses in an expression, evaluate the parts in parentheses <u>first</u>.

<u>Example</u>: Evaluate 8 – (x – 2) with x = -7
<u>Solution</u>: 8 – (x – 2)
8 – (-7 – 2)
8 – (-7 + -2)
8 – (-9)
8 + -(-9)
8 + 9
17

2.2 Exercises:

For each problem <u>write out the original problem</u>, then write out the problem with the substitution of the variable(s), and then show your work as you evaluate the expression.

Evaluate each expression with x = 10 <u>without</u> your calculator.

1. x – 3 – 2
2. 14 – (x – 2)
3. x – (12 – x)
4. (24 – x) – x

Evaluate each expression for x = -3 <u>without</u> your calculator.

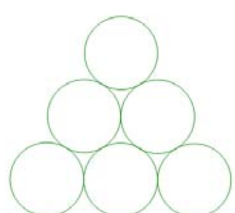
5. x – 3 – 2
6. 14 – (x – 2)
7. x – (12 – x)
8. (24 – x) – x

Evaluate each expression with a = -1 and b = 4 <u>without</u> your calculator.

9. a + 3 – b
10. 14 – (a – b)
11. (a – 3) – b
12. a – b – (b – 2)
13. (a – b) – b
14. (a + 3) – (b – 1)
15. (a – b) – (a – 6)
16. (a + b) – (a – b)
17. b + (17 + b – a)
18. a – 3 – [(b – 1) – a]

19. #Evaluate $\frac{10!}{6!4!}$.

20. * Arrange the first six counting numbers in the circles below so that the sum of the numbers along each side of the triangle is 10.



Assignment #2.3: Absolute Value

The **absolute value** of an integer is the positive distance that the number is from zero. The absolute value of -4 is 4 because -4 is 4 units away from zero. The mathematical notation for "the absolute value of -4" is |-4|.

<u>Example</u>: Find |-20|
<u>Solution</u>: |-20| = 20 because -20 is 20 units away from zero.

<u>Example</u>: Find |0|
<u>Solution</u>: |0| = 0 because 0 is 0 units away from zero.

When simplifying expressions with absolute value signs, the absolute value signs function as grouping symbols.

<u>Example</u>: Evaluate [5 – | -3 – 4|] – | -6 + 1|
<u>Solution</u>: [5 – | -3 – 4|] – | -6 + 1|
[5 – | -7|] – |-5|
[5 – 7] – 5
-2 – 5
-2 + -5
-7

2.3 Exercises:

Find each absolute value.

1. |-3|
2. |12|
3. |-12|
4. |-20|

Find each number described below.

5. 4 units to the right of -1
6. 4 units to the left of -1
7. 4 units to the right of -5
8. 4 units to the left of -5
9. 4 units to the right of the additive inverse of -4
10. 4 units to the right of the opposite of |6|
11. |-3| units to the left of |-1|
12. |-5| units to the left of -5
13. |-3| units to the right of -|-3|

Evaluate

14. |0 · 23|
15. |-4 + -7| + (-5 + 3)
16. -2 + 23 + |-27|
17. -|2 + -7| + -6
18. |8 – 17| – |-4|
19. [-19 – (-15)] + [14 – |-6|]

20. [11 – (-7)] – [(-5 + 2) – (-3)]

21. |12 – 15 | – |0 – 1|

22. Write the following numbers in order from smallest to largest (with the smallest on the left). 2, |-5|, 3, -7, -2, -6, 4, 0, |-6|, -(-2), -100

23. *Which of the following is/are true for <u>all</u> integers? If the statement is true, give the name of the property which guarantees this. If the statement is not true, please provide a counterexample (an example with numbers that shows it not to be true).

a) (x + y) + z = (z + x) + y

b) (x – y) – z = z – (x – y)

c) (x – y) + z = z + (x – y)

Assignment #2.4: Translating into Expressions

Phrases in words can often be translated into a compact numerical or algebraic expression. That's one of the beautiful things about mathematics – complex things can be said in brief and precise notation.

<u>Phrase</u>: 7 more than 23
<u>Translation</u>: 23 + 7

<u>Phrase</u>: 100 decreased by x
<u>Translation</u>: 100 – x

2.4 Exercises:

Evaluate.

1. |-15|

2. |11|

3. |-32|

4. |-(-8)|

5. |-(-(-(2 – 8)))|

6. | 6 – (-(-6))|

7. -| -13 + 4 | + | -(-2) |

8. 1+ [-6 – | | -3 | – 5 |]

9. |-2| – (-11 – 3 + | -5 + 2 |)

Evaluate each expression with a = 15 and b = -2 <u>without</u> your calculator.

10. (a – 3) – (b – 1)　**11.** (a – b) – (a – 6)　**12.** (a + b) – (a – b)

Write as a numerical expression and simplify.

13. the sum of 3 and 4　**14.** 2 less than -30　**15.** -6 increased by 9

16. 14 decreased by 5　**17.** 5 decreased by 14

Write as an algebraic expression.

18. the sum of x and 5 **19.** x less than 23 **20.** x increased by y

21. 14 decreased by x **22.** y decreased by x

Winetka is 12 years old; her birthday is today. Write (and simplify, if possible) an expression for her age:

23. in 3 years **24.** in x years **25.** 4 years ago

26. y years ago **27.** on her birthday in the year 2020

Stunch weighs 100 lbs. Write an expression for his weight if:

28. he gains 20 lbs **29.** he gains x lbs **30.** he loses 18 lbs

31. he loses x lbs **32.** his head (y lbs) and his arm (x lbs) drop off

33. #How many different four-digit numbers can be made from the digits of 1234?

34. *Arrange the digits 1, 2, 3, 4 and 5 (using each exactly once) to form three numbers such that the product of two of the numbers equals the third.

Assignment #2.5: Equations; The Addition and Subtraction Properties of Equality

An **equation** is a statement that two expressions are equal. (An equation with one or more variables is sometimes called an **open sentence**; in this text, we will also call it an equation.) The set of numbers that can be used in place of a variable is called the **replacement set**.

Sadly, not all values from the replacement set make an equation true. However, if one of the values from the replacement set does make an equation true, then that value is called a **solution** of the equation. The set of all solutions of an equation is called the **solution set**.

<u>Example</u>: Solve $x + 27 = 30$ with a replacement set of {1, 2, 3, 4}
<u>Solution</u>: The only value of the replacement set that makes the equation true is $x = 3$.

<u>Example</u>: Solve $x + 27 = 30$ with a replacement set of {5, 6, 7, 8}
<u>Solution</u>: No value of the replacement set makes the equation true, so there is no solution. Our way of showing no solution is to write the symbol **Ø**.

<u>Example</u>: Solve $x + 0 = x$ with a replacement set of {1, 2, 3, 4}
<u>Solution</u>: In this case, all the members of the replacement set make the equation true. So the solution set is {1, 2, 3, 4}.

Solving an equation by checking each of the possible solutions tends to be fairly inefficient, especially if the replacement set is an infinite one. Instead, we employ other methods. One of these methods is to use the **addition property of equality**. This property states that if a = b, then we can add some quantity (like c) to each side of the equation and the equation will still be true. In other words, if a = b, then a + c = b + c.

Why would we ever want to add a value to both sides of an equation?

a. because it's fun
b. because it's really fun
c. because it will help us solve an equation like x – 32 = 15
d. all of the above

Yes, the correct answer is **d**.

<u>Example</u>: Solve x – 32 = 15
<u>Solution</u>: The left hand side is x – 32. If we were to add 32 to both sides, the left hand side would become x, and we would have the solution. Let's do it.

$$\begin{array}{rcl} x - 32 & = & 15 \\ \underline{+\,32} & & \underline{+\,32} \\ x & = & 47 \end{array}$$

Here's where we add 32 to both sides. x – 32 + 32 is just x.
And there is the solution. x = 47.

Check: 47 – 32 = 15

<u>Example</u>: Solve x – 32 = -15
<u>Solution</u>:

$$\begin{array}{rcl} x - 32 & = & -15 \\ \underline{+\,32} & & \underline{+\,32} \\ x & = & 17 \end{array}$$

Again we can add 32 to both sides. x – 32 + 32 is just x.
And there is the solution. x = 17.

Check: 17 – 32 = -15

The **subtraction property of equality** states that if a = b, then we can subtract some quantity (like c) from each side of the equation and it will still be true. In other words, if a = b, then, a – c = b – c.

Why would we ever want to subtract a value from both sides of an equation?

a. because it's fun
b. because it rocks
c. because it will help us solve an equation like x + 32 = 47
d. all of the above

Yes, once again, the correct answer is **d**.

Example: Solve $47 = x + 32$

Solution: The right hand side is $x + 32$. If we were to subtract 32 from both sides, the right hand side would become x, and we would have the solution.

$$\begin{array}{rcl} 47 & = & x + 32 \\ \underline{-32} & & \underline{-32} \\ 15 & = & x \end{array}$$

Now subtract 32 from both sides. $x + 32 - 32$ is just x.
And there is the solution. $x = 15$.

Alternate Solution: Instead of subtracting 32 from each side, we could just add -32 to both sides as follows:

$$\begin{array}{rcl} 47 & = & x + 32 \\ \underline{+ -32} & & \underline{+ -32} \\ 15 & = & x \end{array}$$

Example: Solve $-16 = x - (-7)$

Solution: The right hand side is $x - (-7)$. That is the same as $x + 7$. So we subtract 7 from each side.

$$\begin{array}{rcl} -16 & = & x + 7 \\ \underline{-7} & & \underline{-7} \\ -23 & = & x \end{array}$$

Now we subtract 7 from both sides. $x + 7 - 7$ is just x.
And there is the solution. $x = -23$.

Check: $-23 - (-7) = -23 + 7 = -16$

This is a good time to start learning a few habits during equation solving. These habits will enhance readability and prevent errors.

- Solve an equation in steps shown vertically, not horizontally.
- Keep each successive stage of the equation with the equals signs lined up, each one below the previous one.
- Show explicitly what operation you are doing to each side.
- Check your solution in the original equation.

2.5 Exercises:

Evaluate for $a = 3$ and $b = -2$.

1. $|1 - |3 - a|| + (b - 6)$ **2.** $|b - 5| - |a - 4|$

Write as an algebraic expression.

3. the sum of y and 2   **4.** 15 less than y   **5.** y increased by y

Consider the number line. Find each number described below.

6. 3 units to the left of -13   **7.** 4 units to the right of -5   **8.** 6 units to the left of -17

9. 8 units to the right of -9

10. 12 units to the left of the additive inverse of -3

11. 6 units to the left of the opposite of |-7|

Solve each equation. Format your work as shown in the examples on the previous page and those done in class. Include a check of the solution for at least five problems.

12. $x - 17 = 23$	**13.** $-37 = x - 42$	**14.** $0 = r - 9$
15. $y + -7 = 23$	**16.** $11 = m - 9$	**17.** $p + 17 = 32$
18. $19 = x - (-9)$	**19.** $y + 18 = -40$	**20.** $12 = m - 2$
21. $p + -3 = 19$	**22.** $-21 = x - 0$	**23.** $y - (-1) = 13$
24. $-37 = y - 8$	**25.** $m + 3 = -19$	**26.** $p - 26 = 17$

27. #How many even three digit numbers can be made from the digits 0, 1, 2, and 3 if repetitions are allowed?

28. *A shave and a haircut costs $10. The haircut costs $3 more than the shave. How much does the shave cost?

Assignment #2.6: More Equations to Solve

An equation with an absolute value sign often has more than one solution.

<u>Example</u>: Solve $|x| = 16$
<u>Solution</u>: $x = 16$ or -16

<u>Example</u>: Solve $|a + 2| = 5$
<u>Solution</u>: $a + 2 = 5$ or $a + 2 = -5$

$$\begin{array}{rcl} a + 2 &=& 5 \\ \underline{-2} && \underline{-2} \\ a &=& 3 \end{array} \quad \text{or} \quad \begin{array}{rcl} a + 2 &=& -5 \\ \underline{-2} && \underline{-2} \\ a &=& -7 \end{array}$$

$$a = -7, 3$$

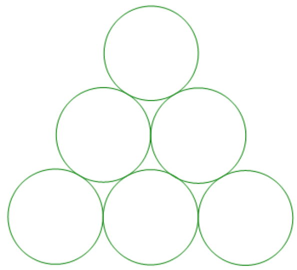
2.6 Exercises:

Solve.

1. $\lvert x\rvert = 3$	**2.** $\lvert x\rvert = 0$	**3.** $19 = x + -22$
4. $26 = y + 6$	**5.** $10 = m - 20$	**6.** $p + 0 = -19$
7. $x - 1 = -28$	**8.** $-2 = \lvert x\rvert$	**9.** $\lvert a - 3\rvert = 2$
10. $-7 = y - 27$	**11.** $m + 0 = 6$	**12.** $-17 = m - (-29)$
13. $\lvert x\rvert = x$	**14.** $\lvert x + 7\rvert = -35$	**15.** $y - (-11) = -31$
16. $m + (-27) = -5$	**17.** $p - 27 = -6$	**18.** $38 = \lvert y + 29\rvert$

19. #How many ways are there to arrange the letters of the word ABRACADABRA?

20. *Arrange the first six counting numbers in the circles below so that the sum of the numbers along each side of the triangle is 12.



Assignment #2.7: Using Equations to Solve Word Problems

Sentences in words can often be translated into a compact numerical or algebraic equation. That's usually the first step in solving a problem with algebra. This assignment is intended to give you practice in the skill of translating situations into equations. So, *even if you can solve the problem without an equation*, and *even if you think that would be faster and altogether the best way ever to solve the problem*, write the equation, just to get the practice. Problems for which you will need equations are headed your way eventually.

Remember our protocol for showing work when solving an equation:

- Solve an equation in steps shown vertically, not horizontally.
- Keep each successive stage of the equation with the equals signs lined up, each one below the previous one.
- Simplify each side of the equation separately before adding or subtracting quantities on each side.
- Show explicitly what operation you are doing to each side.
- Check your solution in the original equation.

Example: 7 more than Kalia's age is 23. Find Kalia's age.

Translation:
$$\begin{aligned} 7 + p &= 23 \\ \underline{-7\quad} & \underline{-7} \\ p &= 16 \text{ years old} \end{aligned}$$

Example: Stunch, who weighs 23 lbs, is 7 lbs lighter than Winetka. Find Winetka's weight.

Translation:
$$\begin{aligned} 23 &= w - 7 \\ \underline{+7} & \quad \underline{+7} \\ 30 &= w \qquad \text{(lbs)} \end{aligned}$$

2.7 Exercises:

Solve each equation.

1. $x - 4 = 23$ **2.** $-4 = x - 42$ **3.** $x - 4 = 4$

4. $33 = r + 33$ **5.** $r + 31 = 3$ **6.** $0 = |w - 1|$

7. $|p + 2| = 38$ **8.** $-4 = w + 11$ **9.** $|w - 1| = -5$

For each exercise below, write and solve an equation to answer the question. Remember to:

a. identify what is being asked for;
b. assign that quantity a variable;
c. write an appropriate equation and solve it;
d. include appropriate units (e.g. years, lbs, etc.) with your solution; and
e. check your answer in the original problem.

10. 17 more than Josephine's age is 23. Find Josephine's age.

11. Pwesco, who weighs 123 lbs, is 17 lbs lighter than Stunch. Find Stunch's weight.

12. The ages of my two children add up to 40. One of them is 13 years old. Find the age of the other one.

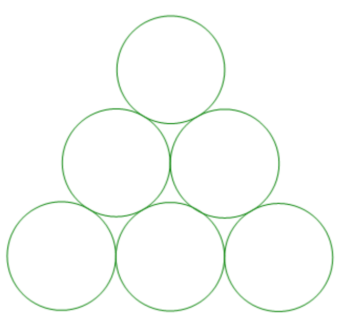
13. 25 less than George's age is 23. Find George's age.

14. Mr. Burgess's pet gerbil, who weighs 43 lbs, is 26 lbs heavier than I. Find my weight.

15. Together, Eric and Matt have $25. Eric has only $16. How much money does Matt have?

16. James finished the marathon 23 minutes before Michael. James took 187 minutes to run the marathon. If they left at the same time, how long did Michael take to finish?

17. * Arrange the first six counting numbers in the circles below so that the sum of the numbers along each side of the triangle is 9.



Chapter 3: Expressions and Equations with Multiplication, Division and Exponents

To work with expressions and equations that have multiplication and division as well as addition and subtraction, we need to agree on the following **order of operations**. (The acronym PEMDAS is often used to help remember the order to follow.)

The standard order in which to do operations is

a. If there are **p**arentheses in an expression, evaluate the parts in parentheses first. If there are parentheses within parentheses (nested parentheses), do the innermost ones first.
b. Then evaluate any expressions that have **e**xponents.
c. Then evaluate any **m**ultiplications and **d**ivisions from left to right.
d. Then evaluate any **a**dditions and **s**ubtractions from left to right.

Note that multiplication and division are at the same level of priority. Similarly, addition and subtraction are at the same level of priority. It is not the case that addition always comes before subtraction or that multiplication always comes before division.

Note also that the fraction bar is a grouping symbol equivalent to parentheses around the numerator and denominator.

Properties of Multiplication:

In the following list, x, y, and z are any integers.

Commutative property — You can multiply in either direction; the result is the same.

$$x \cdot y = y \cdot x$$

Associative property — When multiplying several numbers, you may group as you please.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Identity property — There exists a multiplicative identity (1). This means that there is a number called 1 which can be multiplied by any x to get x itself. (Hence the word *identity*.)

$$x \cdot 1 = x$$

Distributive Property — You may add a group of numbers, then multiply; or multiply to each member of the group, then add. The result is the same. The same is true for multiplication into subtraction.

$$x(y + z) = xy + xz$$
$$x(y - z) = xy - xz$$

The same is true for multiplication from the right.

$$(y + z)x = xy + xz$$
$$(y - z)x = xy - xz$$

Zero property (of multiplication) Zero multiplied by any number is still zero.

$$x \cdot 0 = 0$$

Note: Zero can be divided by any *non-zero* number, and the result will be zero. However, any number (including zero) *divided by zero* is undefined.

$0 \div x = 0$ (if x is not also zero)
$0 \div 0$: undefined
$x \div 0$: undefined

Assignment #3.1: Evaluating Expressions

Example: $22 \div 2 \cdot 11$

Solution: $22 \div 2 \cdot 11$ — Here's the original problem.

$11 \cdot 11$ — Work from left to right (since all we have is multiplication and division). That means $22 \div 2$ comes first.

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Example: $\dfrac{9+15}{^{-}3\cdot 5+3}$

Solution: $\dfrac{9+15}{^{-}3\cdot 5+3}$

$\dfrac{24}{^{-}15+3}$

$\dfrac{24}{^{-}12}$

-2

Example: Evaluate $22 \div {}^{-}(4-2)$

Solution: $22 \div {}^{-}(4-2)$

$22 \div -(2)$

-11

3.1 Exercises:

Evaluate each numerical expression, without your calculator.

1. $8 \cdot -3 - 2$
2. $8 \cdot (3 - 2)$
3. $3 + 2 \cdot -5$
4. $16 - 4 \div -2$
5. $1 + 2(1 + 2)$
6. $7 - 3(5 + -2)$
7. $(8 \cdot 3) - 2$
8. $4 \cdot (5 - 3)$
9. $\dfrac{3+1}{-4+6}$
10. $(5 - 3) \cdot 1$
11. $10 - 6 + 8 \div 2$
12. $-5 \div 0$

13. -30 ÷ (-10 – 5) **14.** (12 + 5) ÷ (-17)

15. $\dfrac{17-(-2-7)}{12+6\div 6}$

16. Insert parentheses to make the following equation true and simplify to show that your solution works:

$$18 + 12 \div 3 = 10$$

17. Insert parentheses to make the following equation true:

$$18 + 12 \div 2 + 3 = 6$$

Suppose we define a new operation @ as follows: a@b = 3a – b.

18. Find 9@2

19. Find -1@5

20. Find 3@(1@-2)

21. Is @ commutative? That is, does a@b = b@a for all real numbers?

22. #Solve | a – 7| = 53.

23. *Nine gears are placed in a loop. Can all the gears rotate simultaneously?

Assignment #3.2: Expressions with Variables

Note: 8x means the same as 8 · x.

Example: Evaluate 8x – 2 for x = -7

Solution: 8x – 2

$8 \cdot {}^{-}7 - 2$

${}^{-}56 - 2$

${}^{-}58$

3.2 Exercises:

Evaluate each expression with x = 10 without your calculator.

1. x · 3 – 2 **2.** 14 · (x – 2) **3.** (24 – 3) · x

4. 24x – x **5.** x · (12 + x)

Evaluate each expression with x = -12 without your calculator.

6. x ÷ 3 – 2 **7.** 14 – (x ÷ 2) **8.** (24 – 3) ÷ (x + 5)

9. 2x ÷ (12 – x) **10.** (24 ÷ x) – x

Evaluate each expression with a = 12 and b = -4 <u>without</u> your calculator.

11. a ÷ 3 – b   **12.** 16 ÷ (a – b)   **13.** b · 12 ÷ a

14. a ÷ b – a ÷ -3   **15.** (a ÷ b) – b   **16.** 1 – 3 · a + a ÷ -b

17. Assign a variable and write and solve an equation to answer this question. Larry has five fewer marbles than Nancy. Larry has 543 marbles. How many does Nancy have?

Insert parentheses to make each of the following equations true:

18. 2 · 9 – 2 · 0 + 8 = 10

19. 2 · 9 – 2 · 0 + 8 = 2

20. 2 · 9 – 2 · 0 + 8 = 34

21. 4 · 3 + 5 – 2 · 4 · 3 = 8

Suppose we define a new operation ☺ as follows: a☺b = ab – 1.

22. Find 3☺-2

23. Find 3☺ (10☺-2)

24. Is ☺ commutative?

25. *My cat always sneezes before it rains. She sneezed today. That means it will start raining soon. Is my logic correct?

Assignment #3.3: Translating into Expressions

<u>Phrase</u>: the product of 7 and 23
<u>Translation</u>: 7 · 23

<u>Phrase</u>: 100 decreased by twice x
<u>Translation</u>: 100 – 2x

3.3 Exercises:

Evaluate each expression for a = -14 and b = 3 <u>without</u> your calculator.

1. (a ÷ 2)(b – 1)   **2.** (2a + 1) ÷ b – (a – 10)

3. (a + 2b) ÷ (a + 4b)   **4.** $\dfrac{|-3-4b|}{9-|a|}$

Write as a numerical expression and simplify. Show your steps.

5. the product of 3 and -4   **6.** 2 less than twice 30   **7.** 5 decreased by the product of 18 and 6

Write as an algebraic expression.

8. the product of x and -5

9. x less than the product of 23 and x

10. x increased by the product of 3 and the opposite of y

11. 14 decreased by the product of x and y

12. twice the result of y decreased by x

Winetka is 12 years old. Write an algebraic expression for how old she will be:

13. when she's twice as old as now

14. when she's x times as old as now

15. two years before she's x times as old as now

16. three years after she's y times as old as now

17. when she is y times her age x years from now

Max weighs x lbs. Write an algebraic expression for:

18. twice his weight

19. 3 lbs less than twice his weight

20. three times the sum of his weight and 12

21. 20 lbs more than his weight, divided by 3

22. the average of Max's weight and Shawna's cat(52 lbs) and Isabella's fish(76 lbs)

23. *A set of dominoes consist of 2 by 1 rectangles, each with 0 to 6 dots on the two squares on each rectangle. All the different possible pairs are part of a set (including doubles) and there are no two matching dominoes. How many dominoes are in a set?

Assignment #3.4: Simplifying Algebraic Expressions

When we have an algebraic expression (i.e. an expression with one or more variables), we can often **simplify** it by replacing it with a simpler equivalent expression. To simplify an expression we use the associative, commutative, and distributive properties, as shown in these examples:

<u>Example</u>: Simplify 6(5y)
<u>Solution</u>: The associative property says that this is equivalent to

$(6 \cdot 5)y$
$30y$

<u>Note</u>: When simplifying we write equivalent expressions vertically without equals signs.

Example: Simplify (5y)6
Solution: The commutative property says that this is equivalent to

$6 \cdot (5y)$ which is precisely the same as the last example.

Example: Multiply 6(x + 5)
Solution:

$6(x + 5)$
$6 \cdot x + 6 \cdot 5$ (distributive property)
$6x + 30$

Example: 6y + 9y
Solution: The distributive property says that this is equivalent to

$(6 + 9)y$
$15y$

Using the distributive property in this way is extremely common. The expressions being added or subtracted are known as **terms**. When they have a common variable factor (y in the example above), it can be factored out of the expression and the remaining parts combined. This process is known as **combining like terms**.

Example: 7a + 9 – 4a + 5
Solution: $7a - 4a + 9 + 5$
$3a + 14$

3.4 Exercises:

Use the associative and commutative properties to simplify the expressions below.

1. $(3x) \cdot 7$ **2.** $26 \cdot 2x$ **3.** $2x \cdot 3y$

4. $2 \cdot (b \cdot 3)$ **5.** $(w \cdot 2) \cdot 5$

Use the distributive property to multiply.

6. $2(1 + x)$ **7.** $3(6 - p)$ **8.** $5(3m - 7)$

9. $(x - 6) \cdot 4$ **10.** $16(2y + 3)$ **11.** $(7 - 3t)4$

Simplify by combining like terms.

12. $2x + 7x$ **13.** $9y - 7y$ **14.** $2y - 8y + 7$

15. $6x + 5 + 9x - 7$ **16.** $15b - 3b - 2 + 5$

Simplify by multiplying and then combining like terms.

17. $2(a + 3) + 5a$ **18.** $(x - 1)9 - 3y$ **19.** $8(y + 6) - 5y + 7$

20. $2(5 + x) - 3x$ **21.** $9(y + 3) - y$ **22.** $6(8 - j) - 5j + -5$

Which of the following are true for <u>all</u> integers? If the statement is true, give the name of the property which guarantees this. If the statement is not true, please provide a counterexample.

23. $(x - y) - z = x - (y - z)$

24. $-(x - y) = -x - y$

25. $x \div x = 1$

26. *All the dominoes in a set are arranged in a chain so that the number of dots on the ends of adjacent dominoes match. If three dots are on one end of the chain, how many dots are on the square at the other end of the chain?

Assignment #3.5: Factoring

Using the distributive property requires care when expressions involve negative numbers. It is important to recall that $-1(x) = -x$ for any integer x. Thus we have the following:

<u>Example</u>: Use the distributive property to simplify $-(x + 3)$.
<u>Solution</u>:
$- (x + 3)$
$(-1)(x + 3)$
$(-1)(x) + (-1)(3)$
$-x + -3$
$-x - 3$

<u>Example</u>: Simplify $-3(2a - 7) - 16$.
<u>Solution</u>:
$-3(2a - 7) - 16$
$(-3)(2a) - (-3)(7) - 16$
$(-3 \cdot 2)a - (-21) - 16$
$-6a + 21 - 16$
$-6a + 5$

Sometimes, we use the distributive property not to simplify an expression, but to write a sum or a difference as a product. Used this way, the process is called **factoring.**

<u>Example</u>: Factor the expression $2x - 16$
<u>Solution</u>:
$2x - 16$
$2x - 2\cdot 8$
$2(x - 8)$

3.5 Exercises:

Use the distributive property to multiply.

1. $12(1 + 3x)$ **2.** $-3(a + 5)$ **3.** $-(k + 3)$

4. $5(-3w - 12)$ **5.** $-1(2y + 3)$ **6.** $-(3j - 1)$

7. $-2(3x + 5)$ **8.** $(-4x - 6)\cdot 4$ **9.** $-5(3x - 7)$

Simplify.

10. $10 - 2(x + 3)$ **11.** $2(6 + m - 6)$ **12.** $20 - (x + 3)$

13. $-5(3p - 7) + 15$ **14.** $-1(-2 - 2x + 3)$ **15.** $-(3x - 1) - 1$

16. $-5 - (5 - y)$ **17.** $(2a - 6)\cdot 4 - 8$ **18.** $-5(-7 - p + 3)$

19. $6p - p(4 - 1) - 2p$ **20.** $-x(x - 5) + 2$

Factor the expressions below.

21. $4x + 4$ **22.** $6x + 12y$ **23.** $20 - 8y$

24. $4p + 24$ **25.** $25 - 5a$ **26.** $51 - 17x$

27. Insert parentheses to make the following equation true and simplify to show that your solution works.

$$18 + 12 \div -2 \cdot 3 = 16$$

28. *In a full set of dominoes how many total dots are there altogether?

Assignment #3.6: The Multiplication and Division Properties of Equality

We can use multiplication and division to solve equations when the variable is being multiplied or divided by a constant.

The **multiplication property of equality** states that if a = b, then we can multiply some quantity (like c) to each side of the equation and it will still be true. In other words, if $a = b$, then $a \cdot c = b \cdot c$.

The **division property of equality** states that if a = b, then we can divide some quantity (like c) into each side of the equation and it will still be true. In other words, if $a = b$, then $a \div c = b \div c$.

Why would we ever want to multiply or divide a value on both sides of an equation?

a. because it's fun

b. because it "rules"

c. because it will help us solve an equation like $32x = 64$ or one like $\frac{x}{5} = 23$.

d. all of the above

Yes, the correct answer is **d**.

Example: Solve $3x = 57$
Solution: The division property of equality says that we can divide both sides by 3.

$$\frac{3x}{3} = \frac{-57}{3}$$
$$x = -19$$

Check: $3 \cdot (-19) = -57$
The 3 being multiplied into x (in the original problem) is known as the **coefficient of x**. When we divide by 3 to solve for x, we are said to be **dividing by the coefficient**.

Example: Solve $\frac{x}{4} = 12$
Solution: The multiplication property of equality says that we can multiply both sides by 4.

$$4 \cdot \frac{x}{4} = 4 \cdot 12$$
$$x = 48$$

Check: $\frac{48}{4} = 12$

3.6 Exercises:

Simplify the following expressions.

1. $3x + 7x$
2. $19y – 2y$
3. $7y – 2y + y – 3y$
4. $5x – 7x + 2$
5. $(-3y – 7) + (2y + 1)$
6. $-1 + 2(3 – 8t)$
7. $2(x + 1) – 6x$
8. $3(y + 2) + 2(1 – y)$
9. $-4(1 – y)$
10. $-(2x + 1) – 3x + 7x$
11. $10 – (7 – 5b)$

Solve and check.

12. $16x = 32$
13. $-36 = 4x$
14. $\frac{x}{7} = -5$
15. $-19x = -38$
16. $19 = -19x$
17. $0 = -4y$
18. $\frac{w}{23} = 1$
19. $-p = 5$
20. $9 = 9r$
21. $12 = \frac{c}{-10}$
22. $0 \cdot r = 33$
23. $\frac{w}{-3} = -20$
24. $15 = -5p$
25. $\frac{x}{17} = 0$
26. $|4y| = 24$

27. Insert parentheses to make the following equation true and simplify to show that your solution works.

$$18 + 12 \div -2 \cdot 3 = -45$$

28. *A frog is at the bottom of a 10 meter well. Each day the frog jumps up 3 meters, but at night it slips back down 2 meters. How many days will it take for the frog to get out of the well?

Assignment #3.7: Numerical Expressions with Exponents

A product of identical factors is called a **power** of that factor. 125 is a power of 5 because 125 is 5 ·5 ·5. When we see a product of factors, we can write it more compactly using exponents. 125 is "5 to the third power", which we write as 5^3. ·

An expression of the form 5^3 is a power. The 5 is called the **base** and the 3 is called the **exponent**.

When simplifying expressions with exponents, it is important to remember to use the correct order of operations. It is also important to note that $-x^a$ means the opposite of x^a, whereas $(-x)^a$ means to raise the opposite of x to the a^{th} power.

Example: Simplify and write in exponential form. $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$
Solution: $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^3 \cdot 5^2 = 5^5$

Example: Write in expanded form and compute. $2^3 \cdot 2^2$
Solution: $2^3 \cdot 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

Example: Write in expanded form and compute. $2^3 + 2^2$
Solution: $2^3 + 2^2 = 2 \cdot 2 \cdot 2 + 2 \cdot 2 = 8 + 4 = 12$

When simplifying expressions with exponents, it is important to remember to use the correct order of operations. It is also important to note that $-x^a$ means the opposite of x^a, whereas $(-x)^a$ means to raise the opposite of x to the a^{th} power.

Example: Evaluate $(-2)^4$
Solution: $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

Example: Evaluate -2^4
Solution: $-2^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$

Example: Perform the indicated operations and simplify. $[11 - (9 - 2)] - 2^2 + 7$

Solution: $[11 - (9 - 2)] - 2^2 + 7$	Here's the original problem
$(11 - 7) - 2^2 + 7$	We do the innermost grouping symbol (9 – 2) first
$4 - 2^2 + 7$	We do the remaining grouping (11 – 7) symbol
$4 - 4 + 7$	Then we do the exponent 2^2
$0 + 7$	Then we subtract, because we are working left to right
7	Then we add

Note that the -2^2 involved squaring the 2 first, then subtracting the resulting 4 from whatever came before it on the left.

3.7 Exercises:

Simplify and write in exponential form.

1. $5\cdot5\cdot5 \cdot 5\cdot5\cdot5\cdot5$

2. $3\cdot3\cdot3 \cdot 3\cdot3\cdot3\cdot3\cdot3\cdot3$

3. $4\cdot4\cdot4\cdot4\cdot4 \cdot 4\cdot4$

4. $(6\cdot6\cdot6\cdot6) \div (6\cdot6)$

5. $y\cdot y\cdot y \cdot y\cdot y\cdot y\cdot y\cdot y$

6. $(9\cdot9\cdot9\cdot9) \div (9\cdot9\cdot9)$

7. $x \cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x$

Write in expanded form and compute.

8. $3^3 \cdot 3^2$

9. $6^4 \div 6^3$

10. $5^3 \cdot 2^3$

11. $2^3 \cdot (2^4)^2$

12. $2^4 \cdot 4^2$

Evaluate.

13. $-5 \cdot -2 \cdot 2$

14. $2 \cdot -12 \cdot -1$

15. $-5 \cdot -12 \cdot -1$

16. $(-3) \cdot (-3) \cdot (-3) \cdot (-3)$

17. $(-3)^4$

18. -3^4

19. $(-2)^5$

20. -2^5

21. $(-1)^3$

22. -1^3

23. $(-1)^{73}$

24. -1^{400}

25. $(-1)^{44}$

26. $4 - 2(4-1)^2$

27. $2(1+3^2)^2 - 1$

28. $(4-2)^3(8-2^3)$

29. $\dfrac{(9-3)^2}{8-2^2}$

30. The operation $\otimes$ is defined as follows: $A \otimes B = \dfrac{A+B}{A-B}$. What is the value of $(6 \otimes 4) \otimes 3$?

31. *A club of 150 members is having a ping pong tournament. When a member loses the game, he or she is out of the tournament. There are no ties. How many games must be played in order to determine the champion?

Assignment #3.8: Variable Expressions with Exponents

3.8 Exercises:

Evaluate each of the expressions below for a = 2, b = -3, c = -7, and d = -1. Remember to write out the problem with the numbers substituted before doing any other steps.

1. ab – d
2. a – 2b
3. abc
4. $3a^2 - 2a + 4$
5. $-2b^2 + 13$
6. $a^2 + b^2$
7. b^3
8. $(a + b)^3$
9. $-(3a + b)^4$
10. 4 – 2(c – b)
11. $-5a^2 - 7a + 2$

Evaluate using the values x = -2, y = 3, and z = -1

12. 2y – 5
13. $5x^2 - 3x + 1$
14. $-y^2 + y - x$
15. $zy^3 - xz$
16. $x(yz + yx)^2 - z$
17. yz + z – y
18. 3yz(x – 2) ÷ xz
19. $-2x^3 + 2x^2 - x + 1$
20. $3z^3 - z^2$

21. *To express 10 as a sum of different powers of 2, you would write $2^3 + 2^1$. The sum of the exponents of these powers is 4. If 100 were expressed as a sum of different powers of 2, what would be the sum of the exponents?

Chapter 4: Solving Equations

Assignment #4.1: Solving Two-step Equations

This is a good time to remind ourselves of our good habits while equation solving.

- Solve an equation in steps shown vertically, not horizontally.
- Keep each successive stage of the equation with the equals signs lined up, each one below the previous one.
- Simplify each side of the equation separately before adding or subtracting quantities on each side.
- Show explicitly what operation you are doing to each side.
- Check your solution in the original equation.

Example: Solve 3x – 8 = -23

Solution:
$$\begin{aligned} 3x - 8 &= -23 \\ \underline{+8} \quad & \underline{+8} \\ 3x \quad &= -15 \\ \frac{3x}{3} &= \frac{-15}{3} \\ x &= -5 \end{aligned}$$

Check: 3(-5) – 8 = -15 – 8 = -23

Example: Solve -17x – 32 = -15

Solution:
$$\begin{aligned} -17x - 32 &= -15 \\ \underline{+32} \quad & \underline{+32} \\ -17x \quad &= 17 \\ \frac{-17x}{-17} &= \frac{17}{-17} \\ x &= -1 \end{aligned}$$

Check: (-17 · -1) – 32 = -15

Example: Solve 1 = |p| – 7

Solution:
$$\begin{aligned} 1 &= |p| - 7 \\ \underline{+7} \quad & \quad \underline{+7} \\ 8 &= |p| \\ p &= \underline{+}8 \end{aligned}$$

Check: |-8| – 7 = 8 – 7 = 1

|8| – 7 = 8 – 7 = 1

Example: Solve |b +2| = 2

Solution: two options:

$$\begin{array}{lcl} b + 2 = 2 & \text{or} & b + 2 = -2 \\ \quad -2 \quad -2 & & \quad -2 \quad -2 \\ b = 0 & & b = -4 \end{array}$$

b = 0, -4

Check: |-4 + 2| = |-2| = 2

|0 + 2| = |2| = 2

4.1 Exercises:

Simplify the following:

1. (3 – 9a)(-1)

2. -4d + 9 + d

3. 7z – 5(3 + z)

4. x – 4(2x + 1) – 3

5. 5(2t) + 4(9 – t) – 37

6. 9(5 + x) – 6(x – 3)

7. 6(x + 2) – 5(x – 1)

8. 7(b + 3) – 2(b + 1)

Solve and check.

9. $3x + 5 = 8$

10. $-2y - 4 = 20$

11. $5z + 7z = -72$

12. $7z + 5 = 5$

13. $7t - 15t = 88$

14. $2a + 3 - 5 = -14$

15. $52 = 13(x + 2)$

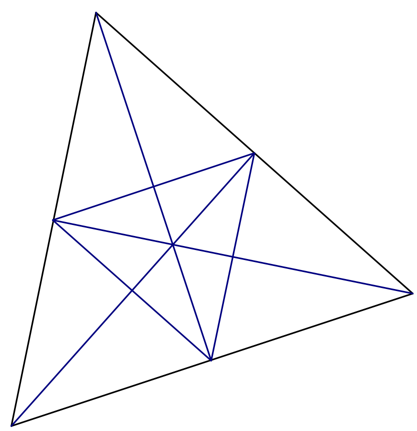
16. $-30x - 9 = -69$

17. $|y| + 3 = 5$

18. $|2g + 1| = 7$

19. #A survey of 30 seventh graders revealed that when the weather gets cold, 16 wear a hat, 18 wear mittens, and 4 wear neither a hat nor mittens. How many students wear both a hat and mittens?

20. *How many triangles are in the figure below?



Assignment #4.2: Solving Multiple-step Equations

Example: Solve $-7x - 31 = x - 15$

Solution: First we decide on a side for the variables. Notice that if we add 7x to each side, the coefficient of the variable will remain positive.

$$\begin{aligned} -7x - 31 &= x - 15 \\ \underline{+7x} \quad &\quad \underline{+7x} \\ -31 &= 8x - 15 \\ \underline{+15} \quad &\quad \underline{+15} \\ -16 &= 8x \\ \frac{-16}{8} &= \frac{8x}{8} \\ -2 &= x \end{aligned}$$

And there is the solution. $x = -2$.

Check: $(-7 \cdot -2) - 31 = 14 - 31 = -17$ and $-2 - 15 = -17$.

Most of the equations we have seen have had exactly one solution, but that is not always the case. Consider the following examples:

<u>Example</u>: Solve $3x - 2 = 3x + 3 - 1$

<u>Solution</u>:	$3x - 2 = 3x + 3 - 1$	Copy the problem
	$3x - 2 = 3x - 2$	Combine the "like term" constants
	$- 3x \quad\quad - 3x$	Subtract 3x from each side
	$-2 = -2$	!!!!

Recall that when we solve an equation we are searching for the values of the variable which make the equation true. Of course, $-2 = -2$ no matter what x is. Thus, any number is a solution to this equation. Such an equation is said to be an <u>identity</u>. We use the symbol $\Re$ (which represents the set of all real numbers) to indicate the solution to this type of equation.

<u>Example</u>: Solve $2x + 1 = 2x - 3$

<u>Solution</u>:	$2x + 1 = 2x - 3$	Copy the problem
	$-2x \quad\quad -2x$	Subtract 2x from each side
	$1 = -3$	!!!!

Of course, we know that $1 \neq -3$ no matter what value x has. Thus, this equation has no solution and we can write the symbol ϕ which represents the null, or empty set.

4.2 Exercises:

Solve and check.

1. $6x + 15 = -33$

2. $0x - 2 = -2$

3. $-30y + (-4) = -694$

4. $2x + 3x + 15 + 3 = 78$

5. $0x + 4 = -7$

6. $|2t - 1| = 5$

7. $3(2z + 4) = 96$

8. $5(3t - 1) = 40$

9. $-5y - 4y + (-3) = 78$

10. $-3p + 27 - 2p = -13$

11. $-5m - 11 = 2m + 45$

12. $-6p + (-23) = p - 23$

Identify a variable, and then write and solve an equation to answer questions 13 and 14.

13. Mia cuts a 6 meter board into two pieces, one twice as long as the other. How long is each of the two pieces?

14. When 18 is subtracted from 6 times a certain number, the result is 96. What was the original number?

15. # In how many distinct arrangements of the letters of BROOM does the B occupy the middle spot?

16. *If at a certain instant it is 7 o'clock in the evening, what time is it exactly 11,999,999,994 hours later?

Assignment #4.3: Using Equations to Solve Word Problems

Example	Corresponding equation
The sum of my weight and 12 equals 212.	y + 12 = 212
The difference between my age and my older brother's age, which is 37 years old, is 15 years.	37 – x = 15
The product of my height and 5 is 300 inches	h · 5 = 300
Three times my dog's age is 51.	3 · y = 51
Eight less than twice my age equals 52.	2y – 8 = 52

4.3 Exercises:

Solve and check.

1. -3x – 22 = 77

2. 8y – 5y = 48

3. 2(y – 4) = -20

4. 8a – 35 = 3a

5. 5z + 7 + 6z + 5 = 45

6. -7(z + 1) + 4 = -31

7. 6x – 15 = 3x + 3

8. -29y + 2 = -5y + 2

Identify a variable, and then write and solve an equation to answer each of the following questions.

9. Ms. Canino paid $344 for an eight performance concert series. What was the price of each performance?

10. Apollo 10 reached a speed of 24,790 miles per hour. That is 37 times the speed of the first supersonic flight in 1947. What was the speed of the 1947 flight?

11. Three times the price of a lollipop plus 20 cents is $9.80; how much is a lollipop?

12. The average of my age and the age 52 is 100 years. How old am I?

13. Aidan cuts a 480 foot wire into three pieces. The second piece is three times as long as the first. The third is four times as long as the second. How long is each piece?

14. *On a twenty question test, each correct answer is worth 5 points, each unanswered question is worth 1 point, and each incorrect answer is worth 0 points. Which of the following scores is impossible to achieve?

90 91 92 95 97

Assignment #4.4: More Multiple-step Equations

Now we can use all the operations and properties we have learned to solve more complicated equations.

Example: Solve $2(3x + 2) + 7 = 35$

Solution:	$2(3x + 2) + 7 = 35$	Copy the original problem
	$6x + 4 + 7 = 35$	Distribute the 2 into $(3x + 2)$
	$6x + 11 = 35$	Add 4 + 7
	$-11 \quad\quad -11$	Subtract 11 from each side
	$6x = 24$	
	$x = 4$	Divide each side by the coefficient 6

Check:
$2(3 \cdot 4 + 2) + 7$
$2(12 + 2) + 7$
$2(14) + 7$
$28 + 7$
35

This is a good time to remind ourselves of our good habits while equation solving.

- Solve an equation in steps shown vertically, not horizontally.
- Keep each successive stage of the equation with the equals signs lined up, each one below the previous one.
- Simplify each side of the equation separately before adding or subtracting quantities on each side.
- Show explicitly what operation you are doing to each side.
- Check your solution in the original equation.

We can also use these techniques to solve equations in which the variable appears on both sides of the equal sign.

Example: Solve $4(x + 2) – 5 = 2(x – 1) + 15$

Solution:	$4(x + 2) - 5 = 2(x - 1) + 15$	Copy the problem
	$4x + 8 - 5 = 2x - 2 + 15$	Use the distributive property
	$4x + 3 = 2x + 13$	Add the constants
	$-2x \quad\quad -2x$	Subtract 2x from both sides
	$2x + 3 = 13$	
	$-3 \quad -3$	Subtract 3 from both sides
	$2x = 10$	
	$x = 5$	Divide each side by 2

Check: We substitute $x = 5$ into each side of the equation:

LHS (left hand side)	RHS
$4(5 + 2) - 5$	$2(5 - 1) + 15$
$4(7) - 5$	$2(4) + 15$
$28 - 5$	$8 + 15$
23	23

4.4 Exercises:

Solve and check.

1. $3(1 - r) + 5r = 2(r + 1)$

2. $5(2 + n) = 3(n + 6)$

3. $3(30 + s) = 4(s + 19)$

4. $5x + 5(1 - x) = x + 8$

5. $4 - 2(a - 2)= -2(a - 3)$

6. $10 + 4(3x - 1) + 2x = 34$

7. $3(2 + c) - 4c = c + 16$

8. $g - 3(5g + 2) = -2(g - 3)$

9. $4(3t - 1) + 13 = 5t + 2$

10. $2(4x + 10) = 3(4x + 12) - 16$

11. $5y + 2(1 - y) = 2(2y - 1)$

12. $3 + |2x - 2| = 25$

Identify a variable, then write and solve an equation to answer these problems.

13. When 28 is subtracted from five times a certain number, the result is 232. What is the number?

14. On Monday, Charlie Brown tried to kick a football several times, only to have Lucy snatch it away at the last moment. On Tuesday, Charlie Brown made twice as many attempts as on Monday, and on Wednesday he tried three times as many times as on Tuesday. If he had 27 futile attempts over the course of the three days, how many times did he try to kick the football on Monday?

15. *Using standard American coins, in how many different ways is it possible to make change for fifty cents?

Assignment #4.5: Even More Multiple-step Equations

4.5 Exercises:

Solve and check.

1. $2(x - 4) + 12 = 20$

2. $1 - 3(x + 1) = -8$

3. $3(2x + 1) + 6 = -33$

4. $5(3x - 1) + 10 = 5$

5. $3(b + 5) - 6 = 3(b + 3)$

6. $2 + 3(3x + 2) - 1 = 16$

7. $5 - 6(2x + 3) = -13$

8. $13(x - 2) = 4(3x - 9)$

9. $6x - 2(2 - x) = 4(2x - 1)$

10. $-3(3x + 4) = 12 - 4(3x + 3)$

11. $4(3x - 2) + x = 3(2x - 4) - 10$

12. $10 - 2(3x + 4) = 4 - 2(2x + 1)$

Identify a variable, then write and solve an equation to answer these problems.

13. Austin G. has a dog and a cat. The dog weighs twice what the cat weighs and altogether they weigh 42 pounds. How much does Austin's cat weigh?

14. At the fruit market you plan to buy two baskets of fruit. Each basket has a nectarine and two pineapples. If a pineapple costs twice as much as a nectarine and you spend a total of $20 on the two baskets, how much does a nectarine cost?

15. *Study the way these pairs of numbers generate new numbers. Then find the values of X and Y.

$(1,2) \rightarrow 5$	$(3,2) \rightarrow 11$	$(3,4) \rightarrow 13$
$(5,1) \rightarrow 16$	$(5,2) \rightarrow 17$	$(6,2) \rightarrow 20$
$\left(\frac{1}{3},1\right) \rightarrow 2$	$(6,3) \rightarrow X$	$(Y,5) \rightarrow 26$

Assignment #4.6: And Still More Equations

4.6 Exercises:

Solve and check.

1. $3(3x - 3) = 3(6x + 6)$

2. $2(2x + 3) + 2 = 5x + 3$

3. $4(3x + 3) = 2(4x - 4) + 12$

4. $3(4x + 4) = 2(3x + 2) + 8$

5. $-3(3x - 3) = 9 - 3(4x + 3)$

6. $4(3x + 4) = 4(3x + 2) + 8$

7. $4(4x + 3) = 2(4x + 9) + 2x$

8. $-2(4x + 4) = 8 - 4(2x + 4)$

9. $-3(3x + 2) = 2 - 2(4x + 4)$

10. $9 + 4(4x - 2) = 3(3x + 2) + 23$

11. $2(2x + 4) = 3(4x - 4) + 36$

12. $4(3x + 2) = 3(4x + 3) - 2$

13. $4x + 4(3x - 2) = 4(4x - 2)$

14. $3(2x + 12) = 4(4x + 13) - 46$

15. $x - 3(3x + 2) = 8 - 4(2x + 4)$

16. $2x + 2(4x + 20) + 5 = 7(4x - 1) + 16$

17. *In how many ways can you arrange the letters of the name of the capital city of Florida while keeping the T as either the first or the last letter?

Table of Contents

Chapter 5: Number Theory

Assignment #5.1: Vocabulary: Multiples, Factors, Divisibility

When two whole numbers are multiplied, the product is said to be a **multiple** of each of the numbers. Thus 42 is a multiple of 7 and a multiple of 6 because $7 \cdot 6 = 42$.

The whole numbers which multiply together to form the product are said to be **factors** of the product. So 6 and 7 are factors of 42. But 3 is also a factor of 42 because $3 \cdot 14 = 42$.

Example: Is 9 a factor of 153?
Solution: Yes, because $9 \cdot 17 = 153$

Example: Find the first five non-zero multiples of 9.
Solution: 9, 18, 27, 36, 45

When a whole number has a factor, that whole number is said to be **divisible** by that factor. So saying that 42 is divisible by 3 is the same thing as saying 3 is a factor of 42. Likewise, 42 is said to be **indivisible** by 5, since 5 is not a factor of 42. (An interesting piece of English language trivia is that the word "indivisibilities" has more "i"s than any other word.)

There are many rules that one can use to quickly establish if one number is divisible by another.

Divisibility by 2
If the number ends in 0,2,4,6 or 8, it is divisible by 2.

Divisibility by 3
If the sum of the digits is divisible by 3, so is the number.

Divisibility by 4
If the last two digits in the number form a new number divisible by 4, then the original number is divisible by 4.

Divisibility by 5
If the number ends in 0 or 5, it is divisible by 5.

Divisibility by 6
If the number is even and divisible by 3, then it is divisible by 6.

Divisibility by 7
Double and subtract the last digit in your number from the rest of the digits. Repeat as necessary. If the result is divisible by seven, then so is the number.

Divisibility by 8
If the last three digits in the number form a new number divisible by 8, then the original number is divisible by 8.

Divisibility by 9
If the sum of the digits is divisible by 9, so is the number.

Divisibility by 10
If the number ends in 0, it is divisible by 10.

Divisibility by 11

Find the sum of every other digit. Then find the sum of the remaining digits. Compute the difference of these two sums. If this difference is 0 or divisible by 11, the number is also divisible by 11.

Example: Find all the factors of 72.
Solution: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

Example: Is 1946 divisible by 3?
Solution: Add up the digits 1 + 9 + 4 + 6 = 20. 20 is not divisible by 3 so 1946 is not divisible by 3.

Example: Is 15 a factor of 657,335?
Solution: This is a place where we would really like to avoid long division. Notice that if 15 is a factor, both 3 and 5 must divide 657,335. While 5 is a factor, 3 is not, so 15 must not be a factor.

Example: Is 18,342,467 divisible by 11?
Solution: Compute 1 + 3 + 2 + 6 = 12. And then compute 8 + 4 + 4 + 7 = 23. Since 23 – 12 = 11 and 11 is divisible by 11, then 18,342,467 is divisible by 11.

Example: Is 270 a factor of 400?
Solution: No. Since 270 is more than half of 400, it cannot be a factor.

Example: Is 15 a factor of 135?
Solution: The only factors of 15 are 3 and 5. If 15 were to be a factor of 135, both 3 and 5 would have to divide 135. Is 135 divisible by both 3 and 5? Yes, since 1 + 3 + 5 = 9 which is divisible by 3 and 135 ends in a 5 which shows it is divisible by 5.

Example: Find all the possible values of n for which the number 3n4 is divisible by 3.
Solution: In these problems the letter n stands for a single digit in the given number. In this case the n is in the ten's place of a number between 300 and 399. We know the number starts with a 3 and ends with a 4; 3 + 4 = 7. To be divisible by 3 the sum of all three of the digits must equal a multiple of 3, so we need single digits for "n" that produce these sums. 3 + 2 + 4 = 9 so 324 works. 3 + 5 + 4 = 12 so 354 works. There is one more digit that works in place of "n". That digit is 8, since 3 + 8 + 4 = 15.

5.1 Exercises:

Do not use a calculator on the following exercises unless a problem is marked with a (c).
List the first five non-zero multiples of each number.

1. 3 **2.** 5 **3.** 7
4. 4 **5.** 13

Is the first number a factor of the second number? Explain your answer mathematically.

6. 3, 75 **7.** 14, 42 **8.** 11, 396
9. 27, 511 **10.** 130, 5200 **11.** 8, 237

Find all the factors of each number.

12. 8 **13.** 12 **14.** 24
15. 142 **16.** 42

17. What is the sum of the positive integer factors of 100?

18. What is the sum of all the possible values of m for which the number 35m is divisible by 2?

19. What is the sum of all the possible values of m for which the number 6m,35m is divisible by 2?

20. List all the factor pairs for 770. Start with 1 and 770.

21. #Solve: $-3(4 - b) = 1 - 4(2 + 5b) - 5$

22. *(c) The middle school basketball team consists of seven players from the 6th, 7th and 8th grades. If the product of their ages is 35,335,872, what is the sum of their ages?

Assignment #5.2: Prime and Composite Numbers; Prime Factorization

A whole number greater than one that has only itself and 1 as factors is called a **prime** number. A whole number greater than one which has more factors than itself and 1 is called **composite**. (One is considered neither prime nor composite. One is simply one.)

The largest known prime as of October 2009 is the Mersenne prime $2^{43112609} - 1$. This number was identified as a prime in August 2008.

Example: Is 51 a prime number?
Solution: The factors of 51 are 1, 3, 17, and 51, so 51 is not prime.

When a number has been expressed as a product of factors all of which are prime, this expression is called the **prime factorization** of that number.

Example: Write the prime factorization of 12 in exponential form.
Solution: $12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$

Example: Write the prime factorization of 24 in exponential form.
Solution: $24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$

One way to arrive at a prime factorization is to divide the number repeatedly by increasing primes.

Example: Write the prime factorization of 120.

Solution: $120 \div \mathbf{2} = 60$

$60 \div \mathbf{2} = 30$

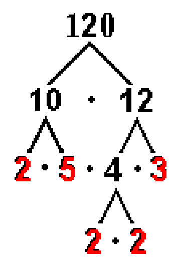
$30 \div \mathbf{2} = 15$

$15 \div \mathbf{3} = 5$

$5 \div \mathbf{5} = 1$

$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^3 \cdot 3 \cdot 5$

Another way to arrive at a prime factorization is to divide the number by any number and then do the same repeatedly with each discovered factor. In this way, one arrives at a "factor tree" of prime factors, the last level of which are all prime factors. The last step is to arrange the factors in increasing order from left to right.



$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^3 \cdot 3 \cdot 5$

Example: If p is a prime other than 2, list all the factors of 4p.

Solution: Note that $4p = 2 \cdot 2 \cdot p$. Hence the factors are 1, 2, p, 4, 2p, and 4p.

5.2 Exercises:

Do not use a calculator for the following exercises.

1. Which shows the prime factorization of 24?

A) $2 \cdot 2 \cdot 3 \cdot 3$

B) $2 \cdot 2 \cdot 2 \cdot 3$

C) $2 \cdot 12$

D) $3 \cdot 8$

2. Which shows the prime factorization of 42?

A) $2 \cdot 3 \cdot 7$

B) $1 \cdot 2 \cdot 3 \cdot 7$

C) $2 \cdot 21$

D) $3 \cdot 24$

3. Which shows the prime factorization of 51?

A) 51

B) $2 \cdot 25\frac{1}{2}$

C) $3 \cdot 17$

D) $3 \cdot 1 \cdot 7$

Write the prime factorization of each number.

4. 77 **5.** 44 **6.** 8

7. 68 **8.** 90 **9.** 21

10. 0	**11.** 54	**12.** 48
13. 104	**14.** 36	**15.** 248
16. 810	**17.** 1701	**18.** 8640

19. Examine some of the prime numbers bigger than three that you have found so far, and find their remainder when divided by 6. (Look at enough so that you can see a pattern.) What do you notice?

20. What are all the values of n that make 546,324,16n divisible by 6?

21. If p and q are distinct odd primes, how many distinct factors does the number 2pq have? For example, 1 and p are both factors. What are the others?

22. A garden has an area of 36 square meters and has sides that are integer lengths. While it isn't likely that the garden is 1 meter by 36 meters, that would have an area of 36 square meters. What are all the other possible dimensions of this garden?

23. List the prime numbers less than 100.

24. Twin primes are pairs of primes that are two apart. Examples include (3, 5) and (5, 7). Find all other sets of twin primes which are less than 100.

25. #Solve $4(t + 2) - 3(6 - 7t) = -10$

26. *Triplet primes are three primes that are two apart. An example is (3, 5, 7). Can you find another set of triplet primes?

Assignment #5.3: Greatest Common Factor

To find the **greatest common factor** (GCF), also sometimes called the greatest common divisor, of two numbers, we look for the largest whole number that is a factor of both numbers. This is sometimes called the greatest common divisor since it gives an integer answer when divided into both numbers.

<u>Example</u>: Find the greatest common factor of 504 and 576

<u>Solution</u>: Put the prime factors of each number in a Venn diagram. The factors in the overlapping part of the diagram will be the factors of the GCF, so multiply them and find the solution. In this case, the GCF is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$.

$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

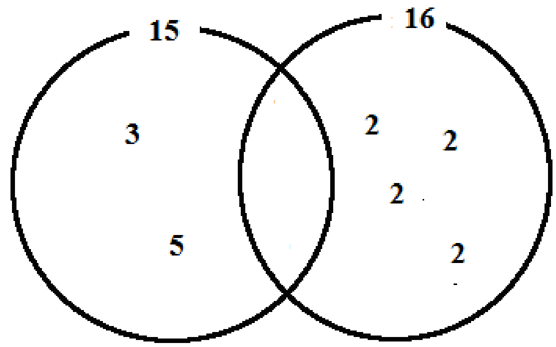
$576 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

<u>Example</u>: Find the greatest common factor of 15 and 16

<u>Solution</u>: Put the prime factors of each number in a Venn diagram. These two numbers have no factors in common, so the GCF is 1. Numbers such as these which do not share any factors other than one are said to be **relatively prime.**

$15 = 3 \cdot 5$

$16 = 2 \cdot 2 \cdot 2 \cdot 2$



<u>Example</u>: Find the greatest common factor of 32 and 40.
<u>Solution</u>: This is a different approach to GCF problems. The technique is called “double division” and it uses two series of divisions to discover common factors. At each step the two numbers are divided by the same selected number, which can be either prime or composite.

2 |32
2 |16
2 |8
2 |4
2

2 |40
2 |20
2 |10
|5

The GCF is composed of those factors that went into both original numbers. In this example it is the first three 2's so the GCF is 2 times 2 times 2 or 8.

5.3 Exercises:

Find the greatest common factor of each set of numbers. For number **10**, you may express your answer as a product of powers of primes. A Venn diagram is not required.

1. 108 and 288 **2.** 81 and 2000 **3.** 300, 400 and 700

4. 450, 500 and 550 **5.** 1080 and 1800 **6.** 1800 and 180

7. 28 and 225 **8.** 480 and 3456 **9.** 48, 72 and 288

10. $2^3 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11$ and $2^5 \cdot 3 \cdot 5^4 \cdot 7 \cdot 13$

11. Use the concept of GCF to answer the following question. What are the dimensions of the largest square tile that can be used to pave a 30 ft. by 42 ft. rectangular patio without any gaps or overlap? Use a sketch to show that your answer is correct.

12. Catherine has a bag of 60 red beads, a bag of 28 green beads and a bag of 48 blue beads. She wants to divide them into piles in such a way that each pile has the same number of each of the three colors of beads.
a) What is the greatest number of piles into which she can divide her beads?
b) How many beads of each color are in each pile that Indigo makes?

13. For what single digit value of n is the number n5,3nn,672 divisible by 11?

14. If p and q are distinct prime numbers, what is the greatest common factor of p and q?

15. #(c) How many different 5-card hands of poker can be dealt?

16. *A **perfect** number is one for which the sum of its proper factors (those factors which are smaller than the number itself) is equal to the number. For example, the proper factors of 4 are 1 and 2. Since 1+ 2 = 3 < 4, 4 is not a perfect number. Find the two smallest perfect numbers.

Assignment #5.4: Least Common Multiple

To find the **least common multiple** (LCM) of two numbers, we look for the smallest whole number that is divisible by both numbers.

Example: Find the least common multiple of 504 and 576
Solution: Put the prime factors of each number in a Venn diagram. The factors in the entire diagram will be the factors of the LCM, so multiply them and find the solution. In this case, the LCM is $7 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2$, which is 4032.

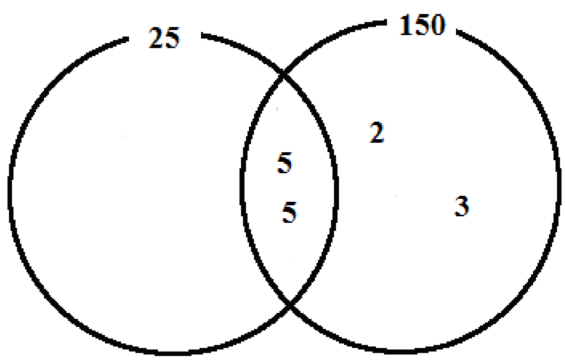
$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

$576 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

Example: Find the least common multiple of 25 and 150
Solution: Put the prime factors of each number in a Venn diagram. Since 150 is already a multiple of 25, the GCF of 25 and 150 is 150.

$25 = 5 \cdot 5$

$150 = 2 \cdot 3 \cdot 5 \cdot 5$



5.4 Exercises:

Find the greatest common factor of each set of numbers.

1. 54 and 144 **2.** 45, 72 and 90

Find the least common multiple of each set of numbers. For number **12**, you may express your answer as a product of powers of primes.

3. 108 and 288 **4.** 2000 and 81 **5.** 300, 400 and 700

6. 450, 500 and 550 **7.** 1080 and 1800 **8.** 1800 and 180

9. 28 and 225 **10.** 3456 and 480 **11.** 48, 288 and 72

12. $2^3 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 11$ and $2^5 \cdot 3 \cdot 5^4 \cdot 7 \cdot 13$

13. Use the concept of LCM to answer the following question. What are the dimensions of the smallest square patio that can be paved by some number of 18 in. by 30 in. tiles without any gaps or overlap? Use a sketch to show that your answer is correct.

14. If p and q are distinct prime numbers, what is the least common multiple of p and q?

15. Grace's pet hedgehog likes to have 48 square feet of area over which to roam. If Grace plans to enclose a rectangular plot with integer length sides in her yard for her hedgehog, how many differently shaped plots are possible?

16. Jack H. walks his dog Bowser around Green Lake every 4 days. Sam walks his dog Bluto around Green Lake every 5 days. Micaela walks her dog Bosco around Green Lake every 2 days. One fine Sunday morning the three friends and their pooches met and walked around the lake together, causing quite a commotion with all of their yapping and barking and canine carryings on. On what day of the week would we expect them all to show up at Green Lake on the same day again?

17. #Solve: $3(x - 5) - (x + 1) = 5x + 2$

18. *A pile of pennies, dimes and half-dollars – 100 coins in all – is worth $5.00. How many dimes are in the pile?

Assignment #5.5: Number Theory Problems

<u>Example</u>: Given the prime factorization $2^3 \cdot 7^5 \cdot 13^2$, what is the other half of this number's factor pair $(2 \cdot 7^2 \cdot 13^2, \_\_)$?

<u>Solution</u>: Since all of the factors are embedded in the prime factorization of a number, each factor pair uses all of the primes. For this number, $2 \cdot 2 \cdot 2 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 13 \cdot 13$, the first factor in the pair uses one of the 2's, two of the 7's and both of the 13's. That leaves $2 \cdot 2 \cdot 7 \cdot 7 \cdot 7$ to make up the second half of the factor pair. So the answer is $2^2 \cdot 7^3$.

5.5 Exercises:

Find both the GCF and the LCM of each set of numbers:

.

1. 24, 60, 132 **2.** 60, 70, 95

3. 72, 126, 198

4. There are three ways to get to and from Fantasy Island, ferry boat, cargo ship, or the *Love Boat.* The ferry boat leaves the island every three days, the cargo ship leaves every four days, and the *Love Boat* leaves every five days. If all three boats

left the island on June 1st, what would be the next date all three boats would leave the island on the same day?

5. For the problem above, if June 1st was a Sunday, what day of the week would it be the next time all three boats left the island on the same day?

6. For what value(s) of n is 234,56n divisible by 4?

7. What is the unit's digit of the number $2 \cdot 5^5 \cdot 7 \cdot 11^5$?

8. Pooja plans to make a rectangular cake to serve 30 people with no leftovers. She estimates that each person will need four square inches of cake. If she has pans in every shape imaginable, but intends to make a rectangular cake with integer length sides, what are the possible cake dimensions that she can bake?

9. What is the sum of all the positive integer factors of 225?

10. What is the other half of the factor pair $(3^2 \cdot 11^5,\ ?)$ for the number $2 \cdot 3^5 \cdot 11^5$?

11. For what single digit value of n is nn6 divisible by 9?

12. Find the GCF and LCM of $9p^2$, 12q and 3pq if p and q are distinct primes greater than 3.

13. #How many different meals can be ordered from a ten item menu if you and your friends want to share five different items? Do not use a calculator.

14. *What is the largest four digit number the product of whose digits is 6! ? How many such four digit numbers are there?

Assignment #5.6: Number Theory Problems

One way to figure out how many factors a number has is to find its prime factorization and build factors from there.
Example: Find the number of factors of 360.
Solution: $360 = 2^3 \cdot 3^2 \cdot 5$. Every factor of 360 is of the form $2^x \cdot 3^y \cdot 5^z$ where x = 0, 1, 2 or 3; y = 0, 1, or 2; and z = 0 or 1. Thus, 360 has $4 \cdot 3 \cdot 2 = 24$ factors.

5.6 Exercises:

1. Michael has two square floors at his house that he intends to cover with tile. He wants to order one kind (color and size) of tile to cover both spaces. One space is 98 inches by 98 inches and the other space is 70 inches by 70 inches. What is the largest square tile that he can use (without cutting any tile) to cover both spaces?

2. What are the different possible integer dimensions (width times height) of a dog run if it must have an area of 360 square feet and be at least 4 feet wide?

3. What is the smallest positive integer that has 8, 30, and 54 as factors?

4. How many different factors does the number 1260 have?

5. How many different factors does the number 16 have? List them all.

6. How many different factors does the number $2^2 \cdot 3^4 \cdot 7^2$ have?

7. Emily is making shelves to store sports equipment and garden supplies in her garage. She would like to make the best possible use of a large sheet of thick plywood that measures 48 inches by 72 inches. How many shelves measuring 12 inches by 16 inches could she cut from the plywood with the minimal amount of waste (assuming the shelves don't lose any of their size during the cutting)?

8. What is the GCF of the numbers whose prime factorizations are $3 \cdot 7^5 \cdot 13 \cdot 23^5$ and $2 \cdot 3 \cdot 7^2 \cdot 23^7$?

9. How many different three digit numbers are there with the property that the product of the digits is 6?

10. #Find n such that $2! \cdot 3! \cdot 4! \cdot n = 8!$

11. *(c) The number 34,459,425 is the product of several consecutive positive odd numbers. What is the greatest of these numbers? (Hint: The best strategy here is not trial-and-error.)

Assignment #5.7: Perfect Squares

5.7 Exercises:

1. What is the prime factorization of 36?

2. What is the prime factorization of 144?

3. What do you notice about the exponents of the prime factors in problems 1 and 2 and what kind of special numbers are 36 and 144?

4. Do you think this pattern of exponents will hold true for all perfect squares? Explain your thinking.

5. If you could multiply the number whose prime factorization is $2^4 \cdot 3^7 \cdot 5^6$ by a single prime, which prime would you choose to make the new number a perfect square?

6. How many different factors does the number 3920 have?

7. Metro bus 351 is scheduled to leave the Northgate transfer station every 42 minutes. Bus 73 is scheduled to leave every 24 minutes. They both leave at 9:00 AM. Assuming they are running on schedule, what is the next time they will both leave the station at the same time?

8. If p is an odd prime number, how many distinct factors does the number 8p have? After you calculate how many factors, list all the factor pairs.

9. If p and q are distinct odd primes, how many distinct factors does the number 4pq have? After you calculate how many factors, list all the factor pairs.

10. The five digit number 4a,ab7 where a and b are both single digits is divisible by nine. What are two possible sets of values for a and b?

11. What is the <u>product</u> of all the values of n that make 234,59n divisible by 11?

12. What is the smallest positive integer that has 8, 30, and 54 as factors?

13. *There are 100 lockers on the first floor of Lakeside Middle School. When the students leave for summer break, all these lockers are carefully closed. Several days into the break, however, Aneesh stops by and decides to open every locker. The next day, Jalen arrives at Lakeside, walks down the hall, and shuts every other locker. On the third day, Jonah goes down the hall and decides to disturb every third locker. Starting with locker number three, he opens every locker which is closed and closes every locker which is open. Similarly, on the fourth day, Olivia disturbs only every fourth locker. If over the course of the summer, 100 students enter the building to open and shut locker doors in this manner, which lockers will be open on the first day of school?

Assignment #5.8: Still More Number Theory Problems

5.8 Exercises:

1. What is the sum of the positive integer factors of 200?

2. 1800 has how many factors?

3. How many factors of 1800 are multiples of 10?

4. What is the smallest positive integer that has 2, 3, 4, 6, 7, and 12 as factors?

5. What is the smallest positive integer n for which 72 is a factor of n! ?

6. How many different four digit numbers are there with the property that the product of the digits is 12? What is the smallest of these?

7. Kalia rolls four standard six-sided dice. The product of the four numbers rolled is 144.

a) Find a possible set of four numbers that Kalia could have rolled.
b) What are all the sets of four numbers that Kalia could have rolled which have this product?

8. The product $a \cdot b = 1200$ and b is an odd number. What is the largest possible value of b?

9. What is the units digit of $6^{10} \cdot 5^{12}$?

10. Suppose that a and b are two positive integers.

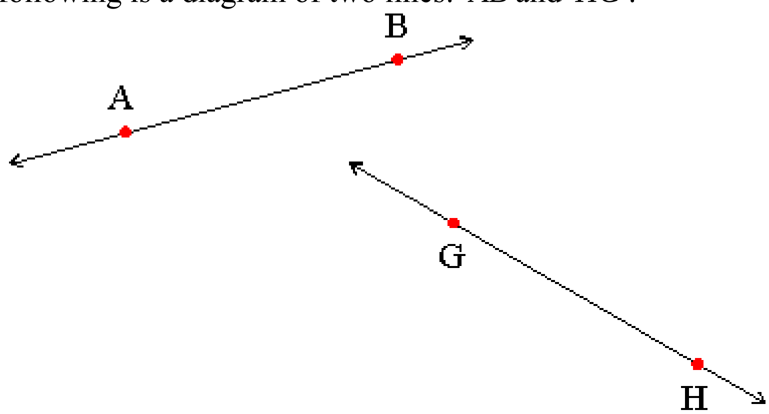
c) If the greatest common divisor of a and b is 100, what is the greatest common divisor of 3a and 3b?
d) Can you say anything about the greatest common divisor of a and 2b?
e) Can you find an example for which the greatest common divisor of a and 2b is still 100?
f) An example for which the greatest common divisor of a and 2b is 200?

11. *Find all six-digit multiples of 22 of the form 5d5,22e where d and e are digits. What is the maximum value of d?

Chapter 6: Lines, Angles, and Polygons

A **line** is a "straight" set of points that we could symbolize by drawing with a ruler on a piece of paper, except that a line extends forever in both directions. We write the name of a line passing through two different points A and B as "line AB" or as $\overleftrightarrow{AB}$.

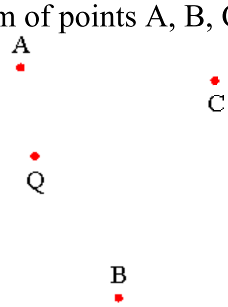
<u>Example</u>: The following is a diagram of two lines: $\overleftrightarrow{AB}$ and $\overleftrightarrow{HG}$.



The arrows signify that the lines drawn extend indefinitely in each direction.

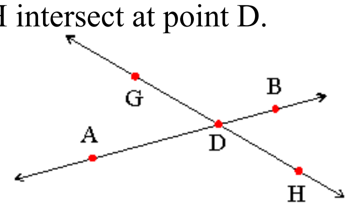
A **point** can be symbolized by a dot on a piece of paper. We identify this point with a number or letter.

<u>Example</u>: The following is a diagram of points A, B, C, and Q:



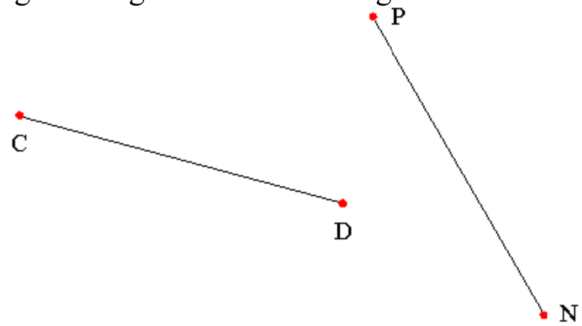
An **intersection** is the place where lines, rays, line segments or figures meet, that is, they share a common point. The point they share is called the point of intersection.

<u>Example</u>: Line AB and line GH intersect at point D.



A **line segment** is a part of a line that does not extend forever, but has two distinct endpoints. A segment with endpoints A and B is called "line segment AB" or $\overline{AB}$.

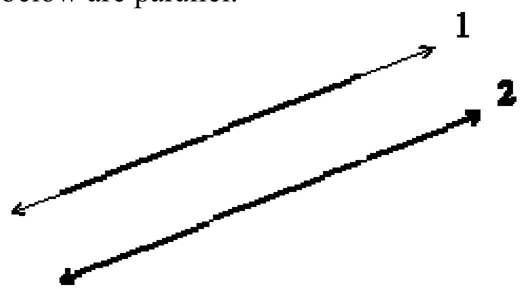
Example: The following is a diagram of two line segments: $\overline{CD}$ and $\overline{PN}$.



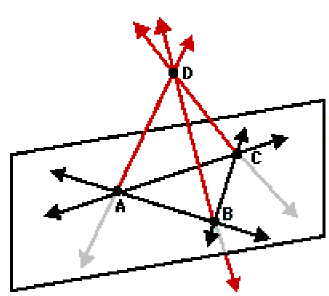
Two lines in the same plane which never intersect are called **parallel** lines. If line 1 is parallel to line 2, we write this as: line 1 || line 2

When two line segments $\overline{EF}$ and $\overline{GH}$ lie on parallel lines, we write this as $\overline{EF} \parallel \overline{GH}$.

Example: Lines 1 and 2 below are parallel.



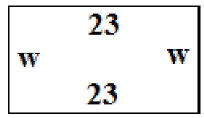
The word **plane** describes a flat surface that goes on forever in all directions. Any three points not on the same line will be in exactly one plane, i.e. the one that contains all three points. In the picture below, A, B, and C are points that determine a plane. D is a point *not* on the plane ABC.



Assignment #6.1: Perimeter and Area of Rectangles

Perimeter means the length of the boundary that encloses some figure, while **area** is the number of square units used to cover the region inside the boundary.

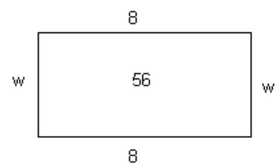
Example: The perimeter of a rectangle is 56 square meters. The length is 23 meters. Find the width.



Solution: Let the width be w. Then 2w + 2(23) = 56. Solve to find w = 5 meters.

NOTE: As you can see from the diagram above, figures in this chapter are not necessarily drawn to scale.

Example: The area of a rectangle is 56 ft^2. The length is 8 ft. Find the width.

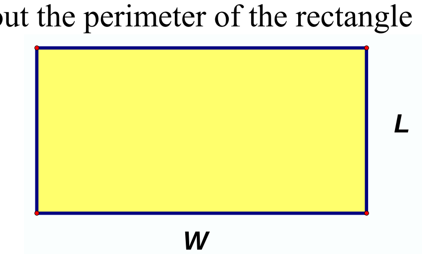


Solution: Let the width be w. Then 8w = 56 so w = 7 ft.

6.1 Exercises:

Please do not use a calculator on the following problems except for those marked with a (c).

1. Fill in each of the blanks in the equations below with the same number to make a true statement about the perimeter of the rectangle shown below.



$$Perimeter = \_\_\_\_\_ \cdot (L + W) = \_\_\_\_\_ \cdot L + \_\_\_\_\_ \cdot W$$

Write and solve an equation to answer each of the following questions.

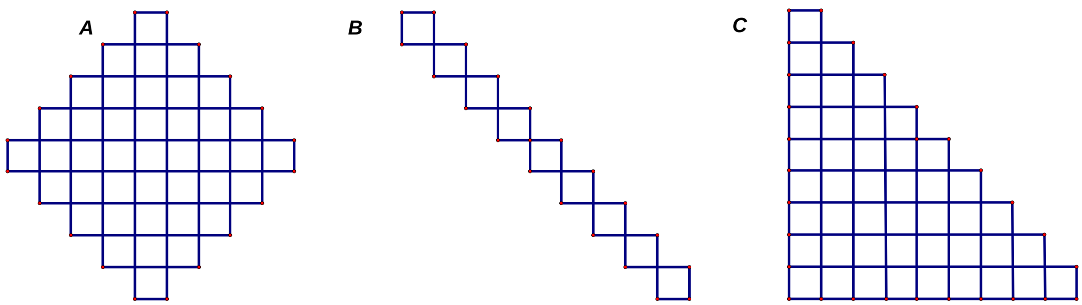
2. The perimeter of a rectangle is 56 cm. The width is 10 cm. Find the length.
3. The perimeter of a rectangle is 234 inches. The length is 60 inches. Find the area.
4. The area of a rectangle is 144 m^2. If the length is 24 m, find the width.

5. The perimeter of a rectangle is 600 meters. If the length is three times its width, find the area.

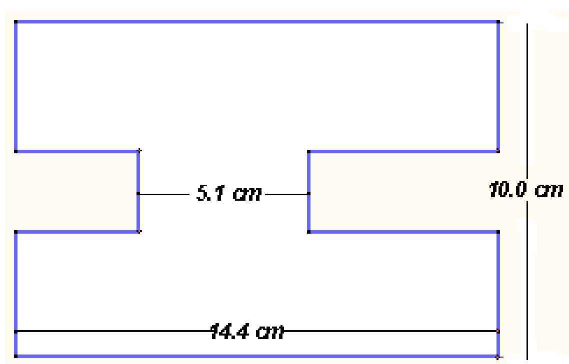
6. The length of a rectangle is 4 cm less than three times its width. If the perimeter of the rectangle is 72 cm, find the area.

7. Matt wants to build a rectangular fence around his vegetable garden at the back of his house. He needs to fence only three sides because the back of the house will serve as one side of the enclosure. Also, he would like to make the side parallel to the house twice as long as the other two sides. If Matt uses 124 meters of fencing, what will be the dimensions of the enclosure? Start by making a sketch.

8. In the figures below, each square is one square unit. The perimeter, therefore, of Figure B is 36 units. Find the perimeter of figures ***A*** and ***C***.



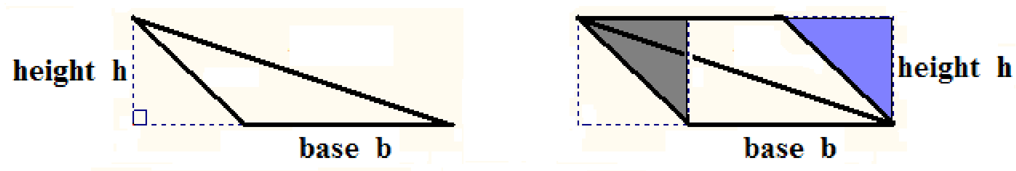
9. Find the perimeter of the polygon below. All angles which appear to be right angles are.



10. *A square game board is seven inches on a side. A square hole one inch on a side is cut from the center. The game board is then cut into four congruent rectangles. How many inches are in the perimeter of each rectangle?

Assignment #6.2: Perimeter and Area of Triangles

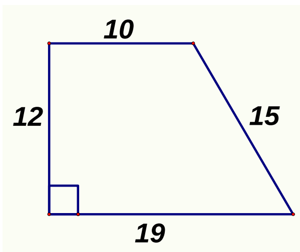
The **area of a triangle** is equal to ½ times the base times the height. This is sometimes written as $A = \frac{bh}{2}$ or $A = ½bh$. The formula can be verified by looking at the figure on the right. Two triangles can be arranged and chopped so as to exactly fit into a rectangle whose dimensions are **b** and **h**. Therefore the area of the triangle must be half of **bh**.



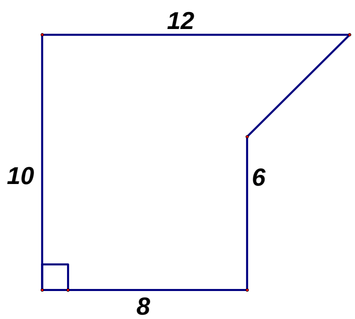
6.2 Exercises:

Please do not use a calculator on the following problems except for those marked with a (c). Each solution should include a labeled sketch and/or an equation. Figures are not necessarily drawn to scale. Angles that appear to be right angles are right angles.

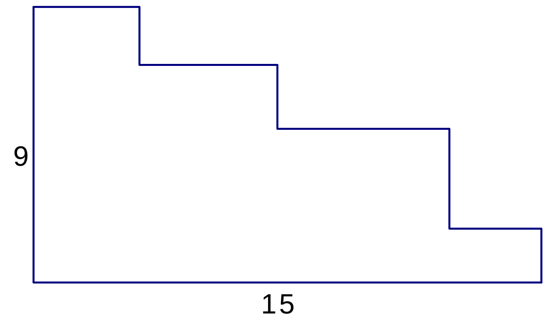
1. Find the area of the figure below.



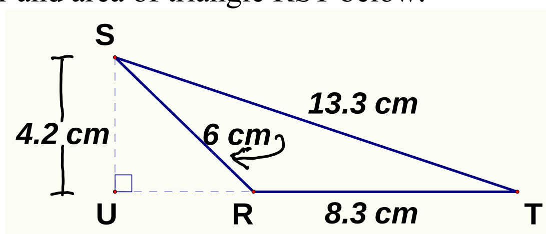
2. Find the area of the figure below.



3. Find the perimeter of the figure below.



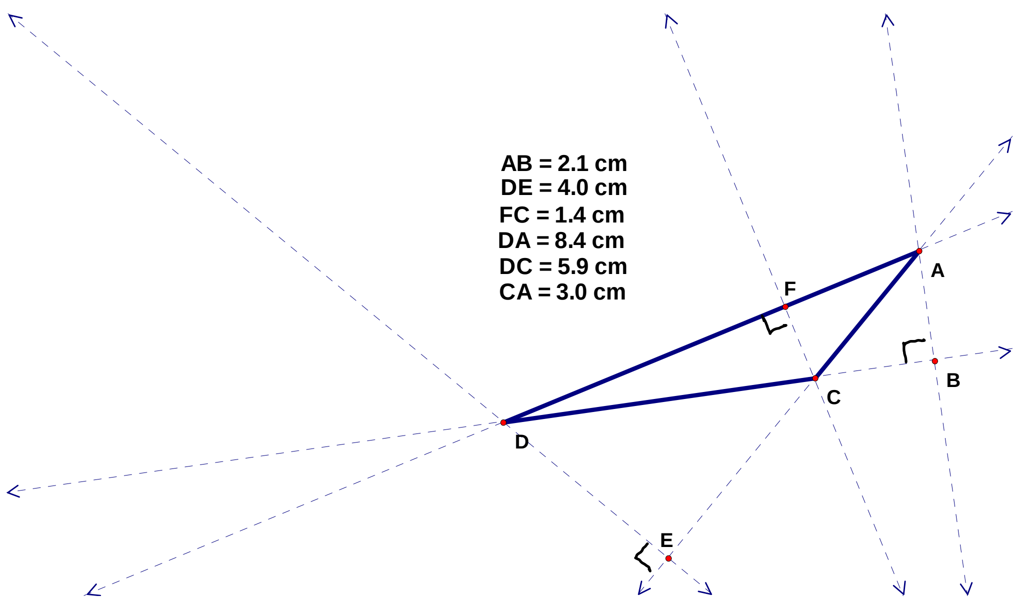
4. Find the perimeter and area of triangle RST below.



5. Write and solve an equation to answer the following question. The perimeter of a triangle is 234 meters. Two of the sides are 31m and 78m. Find the third side.

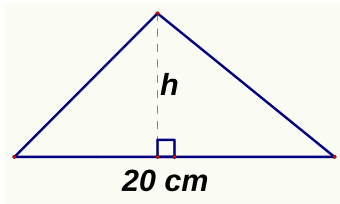
6. (c) At $2.04 per square foot, how much will it cost Snoopy to carpet the floor and two adjacent walls of his doghouse if it is 7 feet by 12 feet and 8 feet high?

7. Find the area of triangle ***ABC*** in the figure below in three different ways. Indicate which lengths you use in your calculations. Round your answer to the nearest integer.



8. ABCD is a square and MNOP are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively. If the perimeter of ABCD is 32 inches, what is the area of quadrilateral MNOP?

9. The base of a triangle with area 84 square centimeters is 20 cm. Find the height, h, of the triangle.



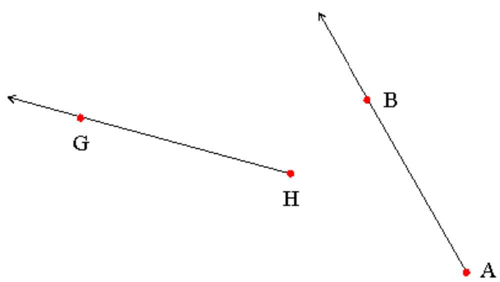
10. Atty's rectangular garden measures 20 meters by 30 meters. It is totally surrounded by a border hedge that is 1 meter wide. What is the ***area*** of the border hedge, in square meters?

11. #How many multiples of 5 are there between 49 and 251?

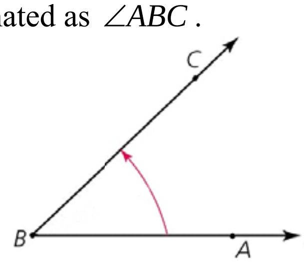
12. *Using a 7-minute sand timer (an old fashioned "hourglass" style timer) and an 11-minute sand timer, can you figure out a simple way to time the boiling of an egg for 15 minutes?

Assignment #6.3: Angles

A **ray** is a half line that begins at a certain point and extends forever in one direction. The point where the ray begins is known as its **endpoint**. We write the name of a ray with endpoint A and passing through a point B as $\overrightarrow{AB}$.

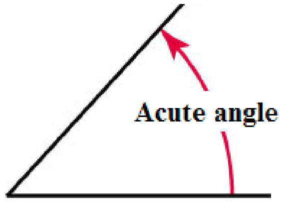


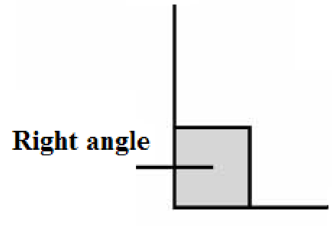
An **angle** is two rays with a common endpoint. In the diagram, B is the common endpoint, and the angle is designated as $\angle ABC$.

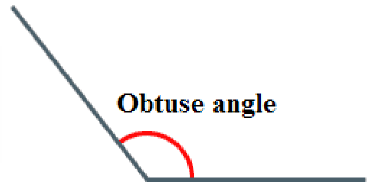


Angles are measured in **degrees**, symbolized with the degree symbol °. Zero degrees (0°) is no angle, 90° is a **right** angle, and 180° is a straight line. An **acute** angle is

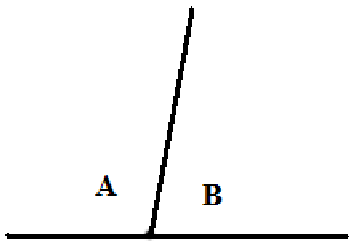
between 0° and 90°, and an **obtuse** angle is between 90° and 180°. A **reflex** angle is greater than 180°. If two lines or line segments meet at a right angle we say they are perpendicular and write $\overline{AB} \perp \overline{BC}$.

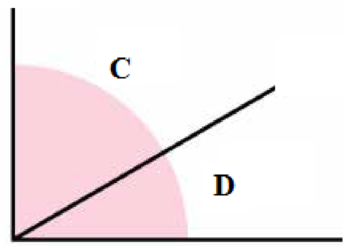






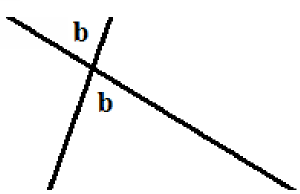
Angles that add up to 90° (C and D below) are called **complementary** and angles that add up to 180° (A and B below) are called **supplementary**. The two angles A and B below form a **linear pair**. Note that any two angles which form a linear pair are supplementary, although the opposite is not necessarily true.

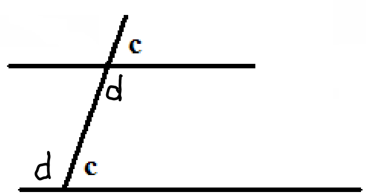


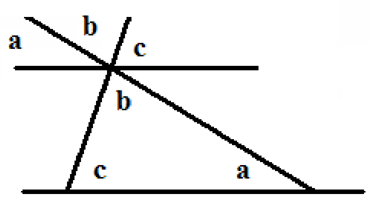


Here are four facts commonly used in geometry:

- When two lines meet, the opposite angles formed are equal in pairs (see angles **b** below, left). Such angles are sometimes called **vertical**.
- When a line crosses two parallel lines, the corresponding angles formed are equal (see angles **c** below, middle).
- When a line crosses two parallel lines, the alternate interior angles formed are equal (see angles **d** below, middle)
- If we apply these two facts to the figure at right below, we can conclude that the angles of a triangle (**a**, **b**, and **c**) add up to a straight line, or 180 degrees.

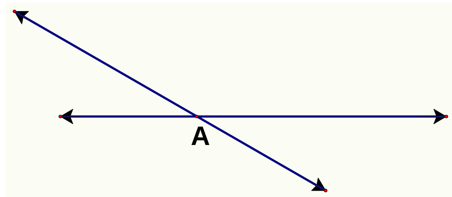




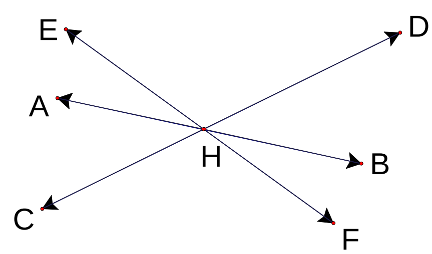


6.3 Exercises:

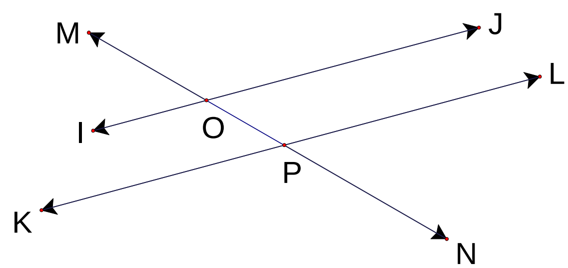
1. In the figure below, two lines intersect in a plane to form vertical angles. How many angle sizes would you need to know in order to determine the size of every angle in this figure?



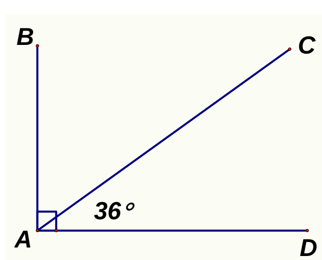
2. How many of these angle measures would you have to know in order to be able to find the measures of the all of the angles?



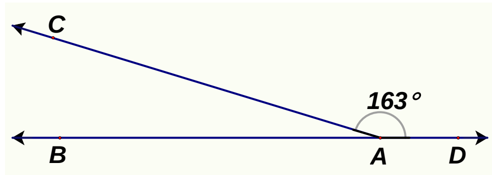
3. In the following figure, eight basic angles are formed at the intersections of the two parallel lines and its transversal $\overline{MN}$. How many of these angle measures would you have to know in order to be able to find the measures of *all* of the angles? Give a short but clear summary of your reasoning.



4. Find the measure of $\angle BAC$ in the figure below.

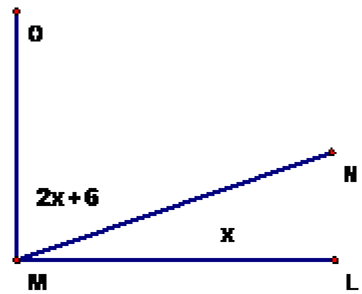


5. Find the measure of $\angle BAC$ in the figure below.



<u>Write and solve an equation</u> to answer questions 6 – 9:

6. Find the measure of $\angle OMN$ in the figure below.

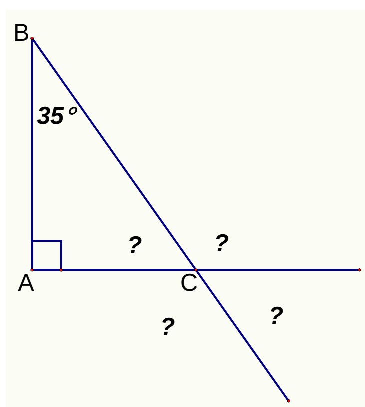


7. Two angles are complementary. If one angle is eight times as large as the other, what is the measure of the larger angle?

8. Two angles form a linear pair. If the larger angle is 6° more than twice the smaller, what is the measure of the larger angle?

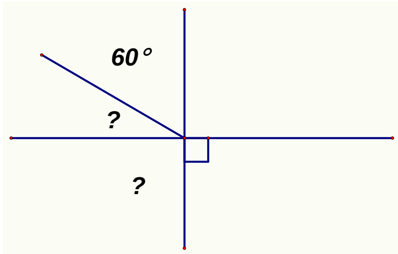
9. Two angles of a triangle are 23° and 49°. Find the third angle.

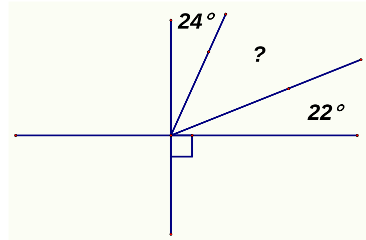
10. Find the size of the unknown angles (those with question marks) in the figure below.



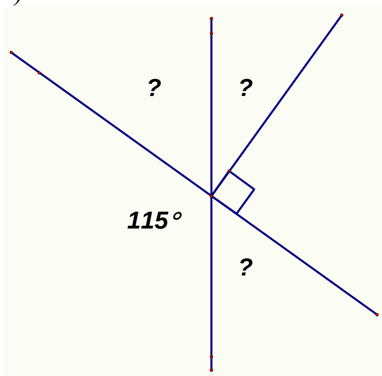
11. Find the measures of the unknown angles (those with question marks) in the figures below.

a) b)

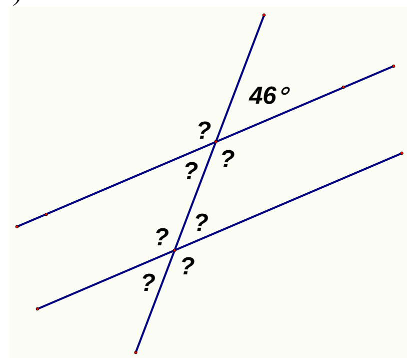




c)

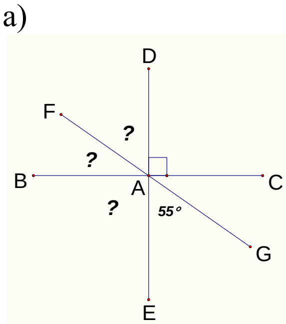


d)

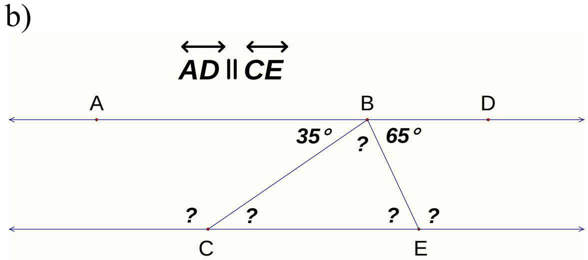


12. Find the measures of the unknown angles (those with question marks) in the figures below.

a)



b)

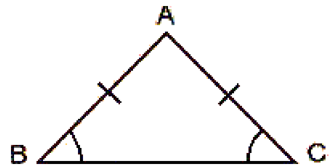


13. *Austin G. tore out several successive pages from *War and Peace*. The number of the first page he tore out was 183, and the number of the last page he tore out used the same digits as 183 in some different order. How many pages did Austin tear out of the book?

Assignment #6.4: Angles in a Triangle

Triangles that have two congruent sides are called **isosceles**. Due to the inherent symmetry of such a triangle, we know that two of the angles are also equal. They are the angles opposite the two equal sides. In the figure below, segments AC = AB; the marks

drawn through those segments are a shorthand for showing that they are equal. Therefore angle B equals angle C.



Triangles that have three congruent sides are called **equilateral**. Due to the inherent symmetry of such a triangle, we know that all three of the angles are also equal, and therefore they are all 60 degrees.

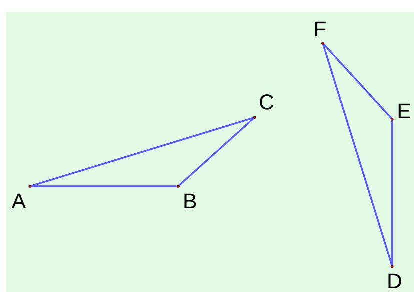
Triangles that have no congruent sides are called **scalene**. Due to the inherent *lack* of symmetry of such a triangle, we know that none of the angles are equal either.

Triangles that have the same size and shape are called **congruent**. Congruent triangles have their corresponding sides and angles equal. If triangle ABC is congruent to triangle DEF, we write this as $\Delta ABC \cong \Delta DEF$.

A triangle with three acute angles is called an **acute triangle**. A triangle with one right angle is called a **right triangle**. A triangle with one obtuse angle is called an **obtuse triangle**.

6.4 Exercises:

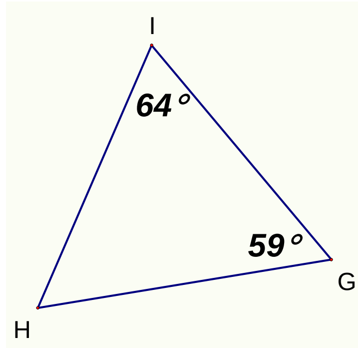
1. Triangle ABC is congruent to triangle DEF. Complete each statement.



$\angle B \cong$ ___________ $\overline{CB} \cong$ ___________

$\angle FDE \cong$ ___________ $\overline{DF} \cong$ ___________

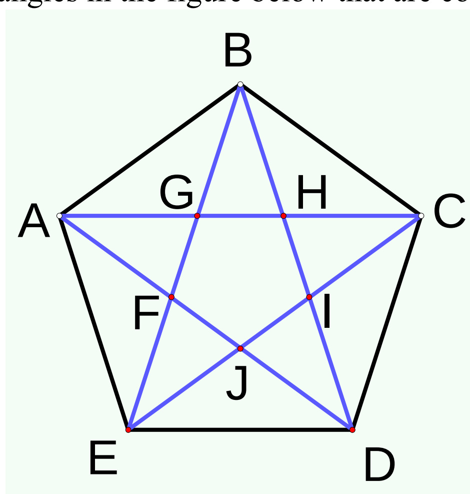
2. In the triangle below, without actually measuring with a ruler, how can you tell which side is the longest? Explain your reasoning.



3. Two angles of a triangle are 123 degrees and 5 degrees. Please write and solve an equation to find the third angle.

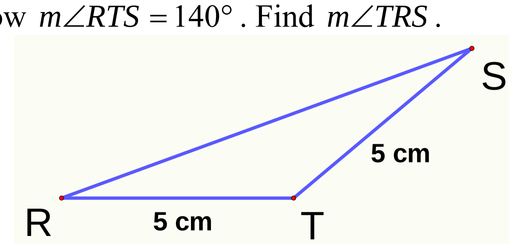
4. Angles A and angle B are vertical angles. If m $\angle A$ = 5x – 4 and m $\angle B$ = 3x + 7, find the measure of the complement of A.

5. Name all of the triangles in the figure below that are congruent to ΔABD.



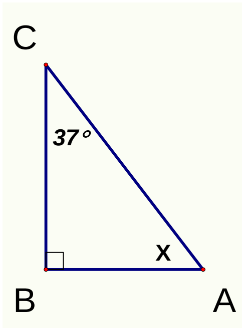
6. From the figure above, name five different triangles, none of which is congruent to any of the others.

7. In the triangle below $m\angle RTS = 140°$. Find $m\angle TRS$.

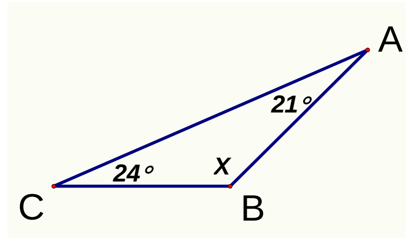


8. Find the measure of each unknown angle in the figures below by writing and solving an equation.

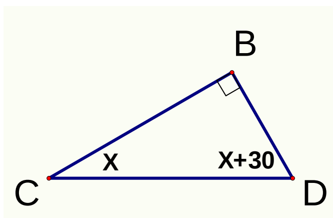
a)



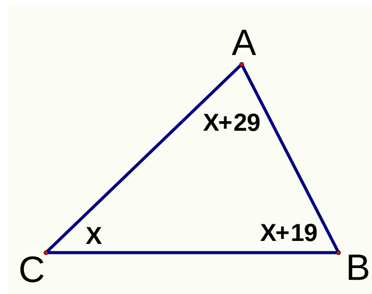
b)



c)



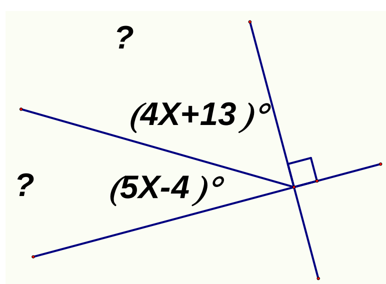
d)



Write and solve an equation to answer questions 8 and 9.

9. In ΔABC, $\angle A$ is 20° larger than $\angle B$, whereas $\angle C$ is 14° smaller than $\angle B$. Sketch and label the triangle and find the measure of each angle.

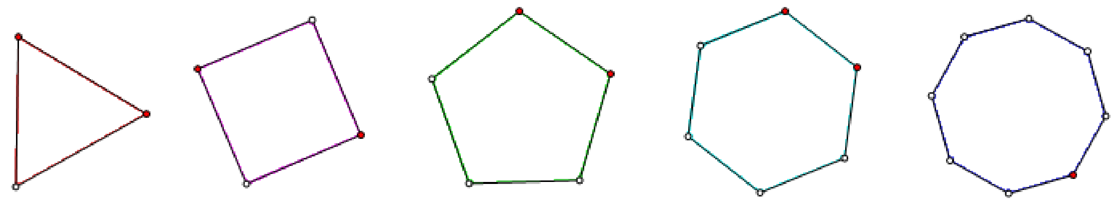
10. Find the measure of the unknown angles (marked with question marks) in the figure below.



11. *How many degrees are in the angle formed by the hands of a clock at 4:00? How about at 4:10?

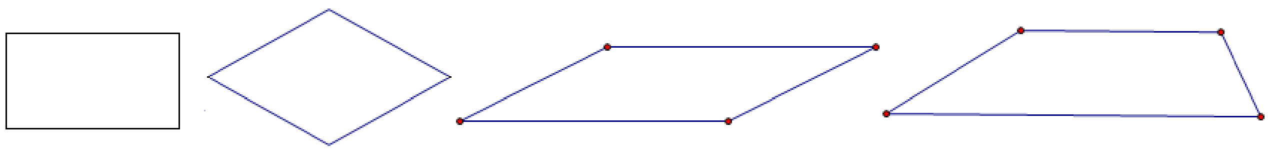
Assignment #6.5: Polygons

A **polygon** is a figure bounded by line segments. Polygons are named according to the number of sides. The standard ones are shown below.



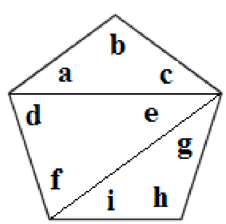
A **regular polygon** is one in which all the sides and angles are congruent. Thus, a regular polygon has symmetry.

Not all polygons are regular. For example, below are several quadrilaterals that are not regular polygons.



A **rectangle** has four right angles. A **rhombus** has four sides of the same length. A **parallelogram** has two pairs of opposite parallel sides. A **trapezoid** has one pair of opposite parallel sides.

The sum of the angles of a polygon with **n** sides (an **n-gon**) is 180(n – 2). For example if n = 3 (a triangle), then the sum of the angles is 180(3 – 2) = 180(1) = 180 degrees. For a quadrilateral, i.e. n = 4, the sum of the angles is 180(4 – 2) = 180(2) = 360 degrees. This formula can be deduced as follows: Below, there is a pentagon with three inscribed triangles. Diagonals drawn from one vertex break the five-sided polygon into only three triangles (n sides makes n-2 triangles). The sum of the pentagon's angles equals the sum of the angles of the three triangles (a+ b + c + d + e + f + g + h + i) or 3(180) = 540.



A polygon with n sides would contain n – 2 triangles in a similar way. And so the sum of the angles would be **180(n – 2)** degrees. (Notice that we did not assume that the polygon is regular, so this applies to all polygons, whether regular or not.)

6.5 Exercises:

Draw each of the following figures.

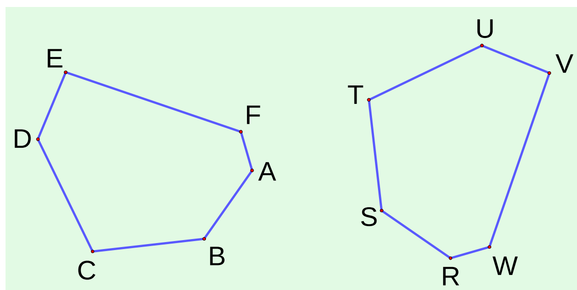
1. A parallelogram

2. An irregular pentagon
3. A scalene triangle

Determine whether each statement is true or false. If it is false, draw a counterexample.
4. Every rhombus is a quadrilateral.
5. Every regular quadrilateral is a square.
6. Every parallelogram is a rhombus.

Find the sum of the measures of the interior angles of each figure described below.
7. An irregular quadrilateral
8. A regular hexagon
9. An obtuse triangle

10. Hexagon ABCDEF is congruent to hexagon RSTUVW. Complete each statement.



$\angle A \cong$ ____________ $\overline{EF} \cong$ ____________

$\angle S \cong$ ____________ $\overline{CD} \cong$ ____________

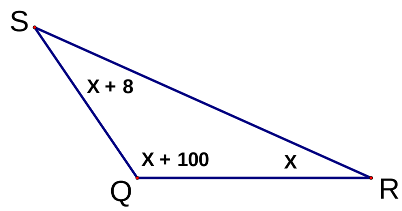
$\overline{RS} \cong$ ____________ $\angle U \cong$ ____________

$\overline{ST} \cong$ ____________ $\angle E \cong$ ____________

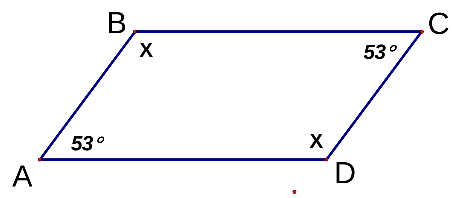
$\angle D \cong$ ____________ $\overline{AB} \cong$ ____________

11. In the figures below, the unknown angles are labeled in terms of x. Write and solve an equation to help you find the measure of each unknown angle.

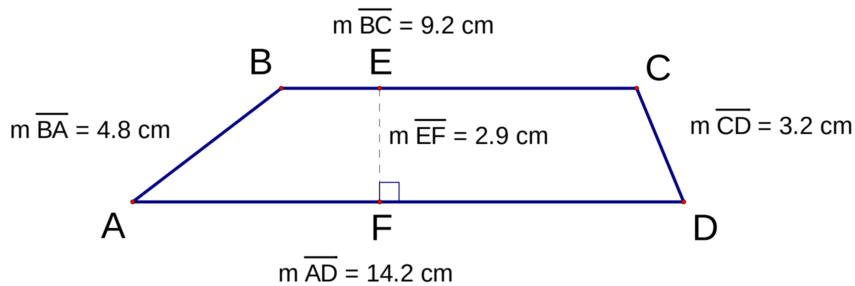
a)



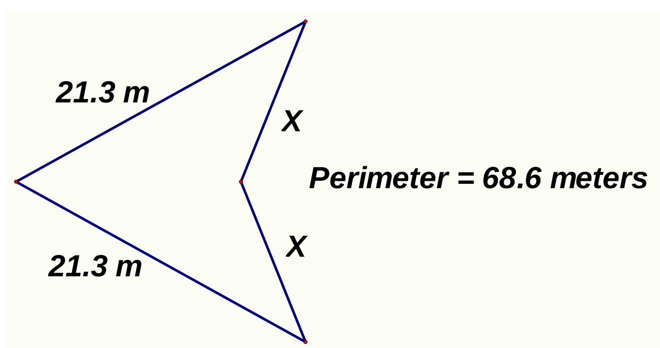
b)



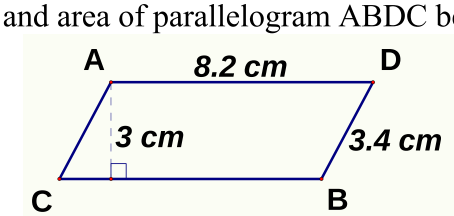
12. Find the area of trapezoid ***ABCD*** in the figure below.



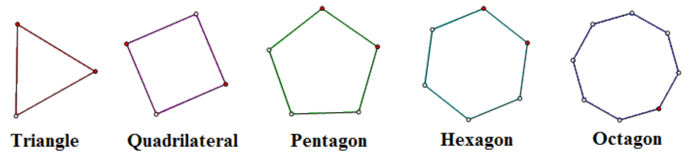
13. Write and solve an equation to find the unknown lengths in the figure below.



14. Find the perimeter and area of parallelogram ABDC below.



15. *The shape of the shut-off valve on a fire hydrant is a regular pentagon. Why would such an unusual shape be chosen? (*Hint: which of the shapes below has no parallel sides?)*



Assignment #6.6: Angles Revisited; Triangle Inequality

Not any three line segments can be used to form a triangle. For example, three sticks of lengths 1 inch, 4 inches, and 6 inches cannot be joined at the ends to form a triangle. More generally, the **Triangle Inequality** states that that for any triangle ABC we have the following three inequalities:

$$AB < AC + BC,\ AC < BC + AB, \text{ and } BC < AB + AC.$$

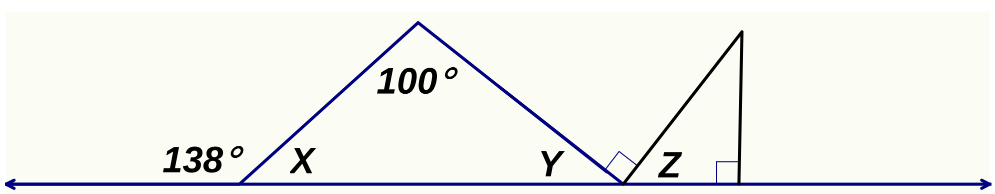
6.6 Exercises:

1. Which of the sets of numbers below could be the lengths of three sides of a triangle?

a) 5, 6, 11
b) 3, 4, 5
c) 4, 9, 11
d) 3, 9, 13
e) -2, 3, 4
f) 7, 12, 21
g) 13, 20, 33

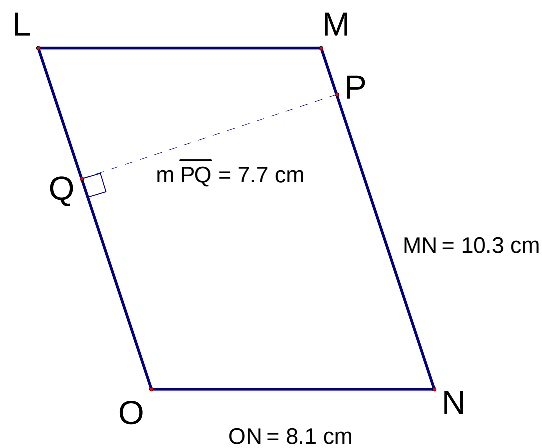
2. Peter found a stick 5 cm long, another 9 cm long, and a third stick **j** cm long. He formed a triangle using the three sticks. What is the sum of all the possible whole number values of **j**?

3. Find the sizes of the angles labeled with variables in the figure below.



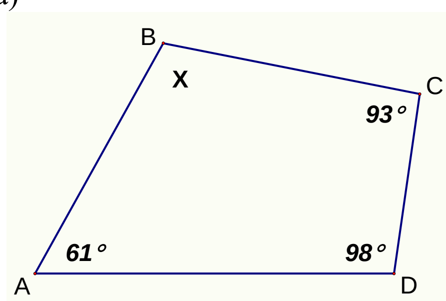
4. In Δ ABC, AC = 3.8 cm and AB = 0.6 cm. If the length of side BC is an integer, what is its length?

5. Find the area of parallelogram ***LMNO*** in the figure below.

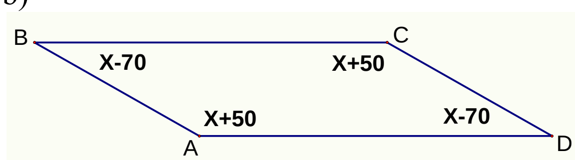


6. In the figures below, the unknown angles are labeled in terms of x. Write and solve an equation to help you find the measure of each unknown angle.

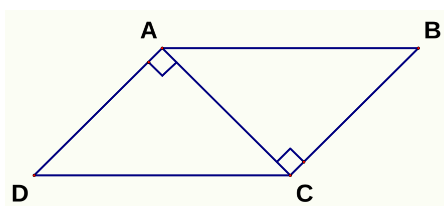
a)



b)



7. In the figure below, diagonal $\overline{AC}$ divides quadrilateral ABCD into two congruent triangles. Refer to this figure to answer the questions below the figure.



a) Which segment is perpendicular to $\overline{AD}$?

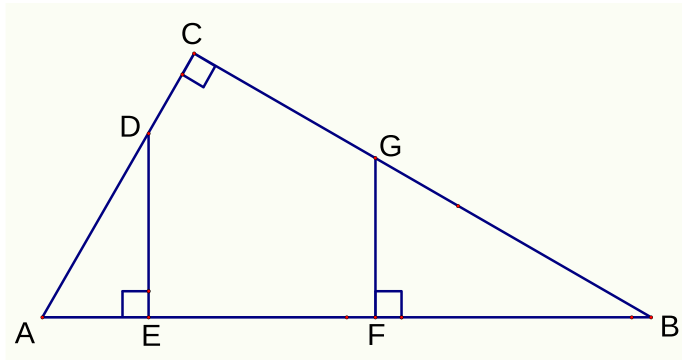
b) Which segment appears to be parallel to $\overline{AD}$?

c) If the perpendicular sides of ΔACD are 6 cm and 6 cm, what is the area of ΔACD?

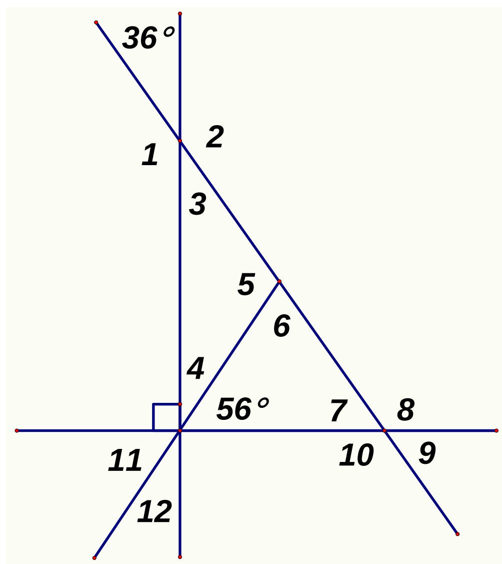
d) What is the area of ΔCAB?

e) What is the area of quadrilateral ABCD?

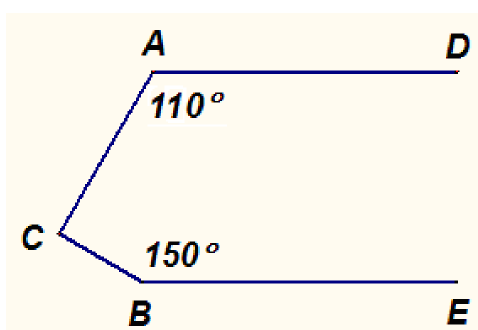
8. In the figure below, m$\angle ADE = 30°$. Find the measure of $\angle CGF$.



9. Find the measures of the numbered angles in the figure below.



12. *In the figure below, $\overline{AD} \| \overline{BE}$. Also, $m\angle CAD = 110°$ and $m\angle CBE = 150°$. Find $m\angle ACB$.



Assignment #6.7: Diagonals of a Polygon

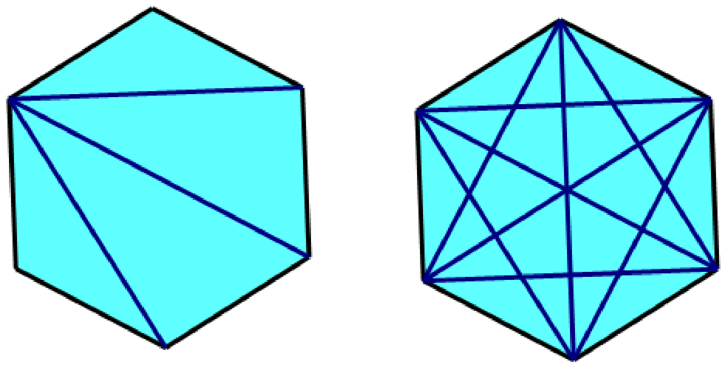
Example: Find the number of diagonals in a hexagon.

Solution: Consider the diagram below. Note that from each of the six vertices of the hexagon, there are three diagonals. Drawing three diagonals from each of the 6 vertices

leads to (6)(3) = 18 segments. However, since this method would lead to each diagonal being drawn twice, once for each end, we must divide by two, leading to

Number of diagonals = $\frac{(6)(3)}{2} = 9$.

<u>Alternate solution</u>: Each diagonal is described by choosing its two endpoints. There are six choices for the first endpoint, and three choices for the second. Divide by two since each segment is counted once from each end. This leads to the same calculation as above.



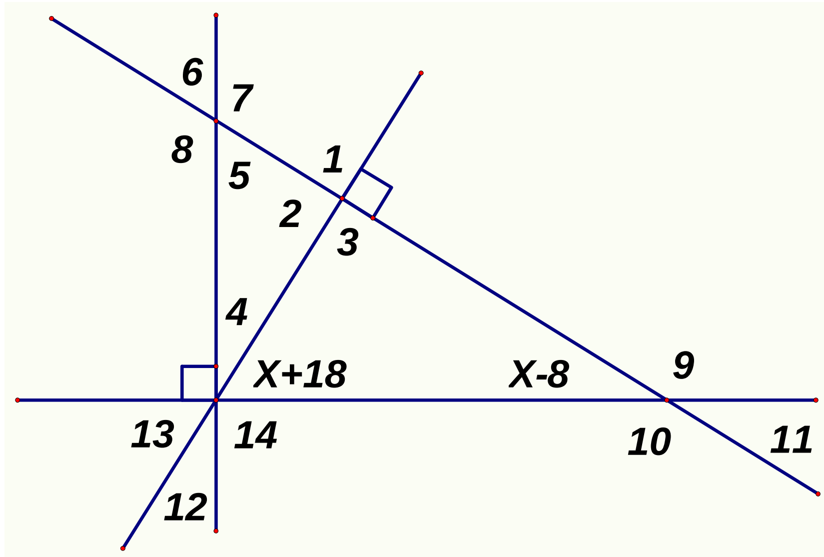
6.7 Exercises:

1. How many diagonals can be drawn from <u>one</u> vertex of a regular pentagon? (Note: Draw a picture.)
2. How many diagonals can be drawn from <u>one</u> vertex of a convex 10 sided polygon?
3. How many distinct diagonals (total) can be drawn in a regular pentagon?
4. How many distinct diagonals (total) can be drawn in a convex octagon?
5. How many distinct diagonals (total) can be drawn in a convex polygon with **n** sides?

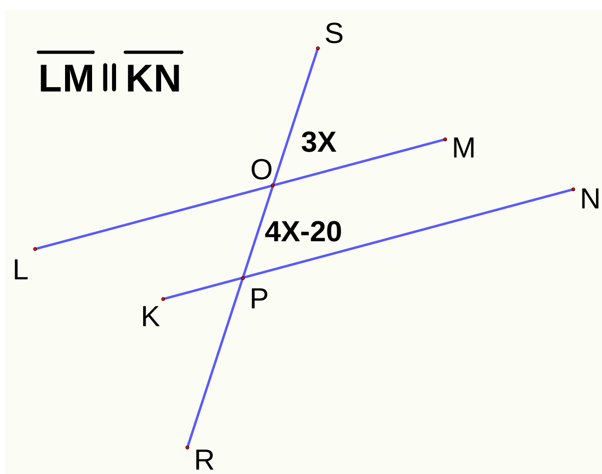
Find the measure of a single angle of each of the following.

6. A square
7. An equilateral triangle
8. A regular octagon
9. A regular hexagon
10. A regular decagon

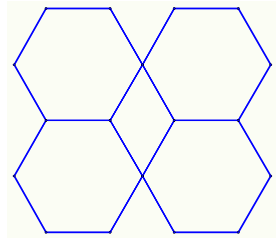
11. Find the measures of the numbered angles in the figure below.



12. Find m$\angle LOP$ and m$\angle NPR$ in the figure below.



13. A tessellation is composed of four regular hexagons and a rhombus as shown. How many degrees are in an acute angle of the rhombus?



14. * In the country of Gaussland, the distance from Pascalidonia to Erdöston is 660 miles, from Pascalidonia to Fermat Heights is 310 miles, from Fermat Heights to Galois City it is 200 miles, and from Galois City to Erdöston it is 150 miles. How far is it from Fermat Heights to Erdöston?

Chapter 7: Rational Numbers

The numbers on the number line can be divided into the rational numbers and the irrational numbers. Rational numbers are those that can be expressed as a ratio of two integers (positive or negative). In other words, a **rational** number can be expressed as a fraction, with integers in the numerator and denominator. **Irrational** numbers cannot be expressed in this way.

Examples of rational numbers are $2, -1.2, \frac{2}{7}$, all integers, all terminating decimals, and all repeating decimals

Examples of irrational numbers are $\sqrt{3}$, $-\sqrt[3]{5}$, π

Assignment #7.1: Fraction Skills: Reducing Fractions, Converting between Improper Fractions and Mixed Numbers

The symbol $\frac{x}{y}$, where x and y are whole numbers and $y \neq 0$, is called a **fraction**. It is numerically equal to $x \div y$. Thus $\frac{6}{2} = 6 \div 2 = 3$. Likewise $\frac{4}{8} = 4 \div 8 = \frac{1}{2}$.

The fraction bar is called the **vinculum**, and the number above it is the **numerator** and the number below it is the **denominator**.

If you multiply or divide the numerator and denominator of a fraction by the same (non-zero) number, you get an **equivalent fraction**. This can be stated symbolically as $\frac{x \cdot a}{y \cdot a} = \frac{x}{y}$ and $\frac{x \div b}{y \div b} = \frac{x}{y}$, provided $y \neq 0$ and $a \neq 0$ and $b \neq 0$.

Example: Reduce this fraction to lowest terms. $\frac{504}{576}$

Solution: The GCF of 504 and 576 is 72. Divide the top and bottom by 72, and the result is $\frac{7}{8}$.

Alternate solution: Divide both the numerator and denominator by common factors.

$$\frac{504}{576} = \frac{252}{288} = \frac{126}{144} = \frac{63}{72} = \frac{7}{8}$$

Note that the rational numbers obey the usual properties of addition and multiplication: commutative, associative, identity, inverse, and distributive.

In particular, the opposite of any fraction may take several forms. For example, $\frac{-7}{8}$, $\frac{7}{-8}$, $-\frac{7}{8}$, -0.875 are all ways to indicate the opposite of $\frac{7}{8}$.

Fractions can have three different types:

Proper Fractions: The numerator is less than the denominator

Example: $\frac{35}{45}$

Improper Fractions: The numerator is greater than the denominator

Example: $\frac{78}{48}$

Mixed Fractions (or **Mixed Numbers)**: A whole number and proper fraction together

Example: $1\frac{2}{3}$ [which means $1 + \frac{2}{3}$]

Important note: In general, when we refer to a reduced fraction, we mean a proper or improper fraction in lowest terms.

To convert an improper fraction to a mixed fraction, follow these steps:

1. Divide the numerator by the denominator.
2. Write down the whole number answer.
3. Then write down any remainder above the denominator.

Example: Convert $\frac{78}{48}$ to a mixed fraction.

Solution: Divide 78 by 48, which is 1 with a remainder of 30

Write down the 1 and then write down the remainder (30) above the denominator (48).

Then reduce $\frac{30}{48}$ to $\frac{5}{8}$. The final result is $1\frac{5}{8}$.

To convert a mixed fraction to an improper fraction, follow these steps:

1. Multiply the whole number part by the fraction's denominator.
2. Add that to the numerator.
3. Then write the result on top of the denominator.

Example: Convert $1\frac{5}{8}$ to an improper fraction.

Solution: Multiply the whole number 1 by the denominator (8). Then add the numerator (5) to get 13. Then put this over the denominator. The final result is $\frac{13}{8}$.

7.1 Exercises:

On this assignment, do each problem without a calculator, and then use your calculator to check. Express all answers as reduced fractions unless otherwise directed.

Reduce each fraction to lowest terms.

1. $\frac{35}{45}$ **2.** $\frac{135}{345}$ **3.** $\frac{18x}{81x}$

4. $\frac{39}{21}$ **5.** $\frac{78}{48}$ **6.** $\frac{56xy}{80y}$

7. $\frac{104}{286}$ **8.** $\frac{49ab}{85abc}$ **9.** $\frac{42}{54}$

Convert to a mixed fraction.

10. $\frac{60}{40}$ **11.** $\frac{225}{175}$ **12.** $\frac{493}{473}$

13. $\frac{351}{162}$ **14.** $\frac{408}{391}$ **15.** $\frac{630}{390}$

16. $\frac{84}{72}$ **17.** $\frac{225}{221}$ **18.** $\frac{570}{90}$

19. $\frac{170}{102}$ **20.** $\frac{120}{32}$ **21.** $\frac{84}{32}$

Convert to an improper fraction.

22. $3\frac{5}{12}$ **23.** $5\frac{4}{7}$ **24.** $3\frac{26}{30}$

25. $4\frac{6}{13}$ **26.** $3\frac{22}{24}$ **27.** $2\frac{20}{21}$

28. $2\frac{18}{10}$ **29.** $5\frac{8}{25}$ **30.** $4\frac{27}{22}$

31. $6\frac{18}{24}$ **32.** $9\frac{15}{21}$

33. *At a wishing fountain last Friday, one fourth of the people tossed in a quarter, 40 people tossed in a dime and the rest tossed in a penny. In total, 200 coins were tossed into the fountain. How much money was thrown into the fountain that day?

Assignment #7.2: Converting between Decimals and Fractions

Every repeating or terminating decimal can be expressed as the ratio of two integers, and vice versa.

Example: Convert each of the following to a decimal (either repeating or terminating):

a. $\frac{4}{5}$

b. $-11\frac{1}{3}$

c. $1\frac{3}{5}$

Solution: Just divide the numerator by the denominator, and express the result as a decimal. Alternatively, express the fraction whose denominator is a power of 10.

a. $\frac{4}{5} = 0.8$

b. $-11\frac{1}{3} = -11.\overline{3}$

c. $1\frac{3}{5} = 1\frac{6}{10} = 1.6$

Example: Prove that $1.\overline{7}$ is rational by expressing it as the ratio of two integers in the form $\frac{a}{b}$.

Solution: Let $x = 1.\overline{7}$

$$\begin{aligned} \text{Then } 10x &= 17.\overline{7} \\ \underline{-x} &= \underline{-1.\overline{7}} \\ 9x &= 16. \\ x &= \frac{16}{9}. \end{aligned}$$

Thus, $1.\overline{7} = \frac{16}{9}$, which is rational because it's the ratio of two integers.

Example: Convert 0.24, -1.8, and $1.\overline{7}$ to fractions (with numerator and denominator being integers):

Solution:

$$0.24 = \frac{24}{100} = \frac{6}{25}$$

$-1.8 = -\frac{18}{10} = -\frac{9}{5}$

$1.\overline{7} = 1\frac{7}{9} = \frac{16}{9}$

7.2 Exercises:

Do each problem below without a calculator and then use your calculator to check.

Prove that each number below is rational by expressing it as the ratio of two integers in the form $\frac{a}{b}$.

1. 78 **2.** 2.7 **3.** -1.4

4. $0.\overline{2}$ **5.** $-0.\overline{5}$ **6.** 0.32

7. $3.\overline{5}$ **8.** $-2.\overline{2}$ **9.** $0.\overline{54}$

10. 23.2 **11.** $23.\overline{23}$ **12.** $3.\overline{2}$

13. $4.\overline{34}$ **14.** $2.\overline{324}$ **15.** -5

16. -0.05 **17.** $4.5\overline{4}$

Convert to a decimal (either repeating or terminating).

18. $\frac{3}{10}$ **19.** $\frac{23}{100}$ **20.** $\frac{4}{5}$

21. $\frac{1}{8}$ **22.** $\frac{1}{3}$ **23.** $\frac{5}{9}$

24. $\frac{1.4}{2.1}$ **25.** $\frac{135}{35}$ **26.** $\frac{42}{54}$

27. $\frac{14a}{8a}$ **28.** $\frac{18.5}{108}$ **29.** $\frac{1.28}{0.2}$

30. *You have probably noticed that some fractions terminate when written as decimals and some don't. Did you notice that all the fractions that don't terminate have a common characteristic? Develop a rule to determine if a fraction written as a decimal will become a repeating decimal or if it will terminate.

Assignment #7.3: Comparing Rational Numbers

We can compare rational numbers in several ways.

a. We can express the fractions with the same denominators and then compare the numerators; or

b. We can express them with the same numerators and compare the denominators; or
c. We can compare their cross-products; or
d. We can subtract one from the other and see if the result is positive or negative; or
e. We can compare them to a familiar fraction.

Example: Compare $\frac{3}{7}$ and $\frac{1}{2}$.

Solution: $\frac{3}{7}$ is $\frac{6}{14}$ and $\frac{1}{2}$ is $\frac{7}{14}$. Since $6 < 7$, it follows that $\frac{3}{7} < \frac{1}{2}$.

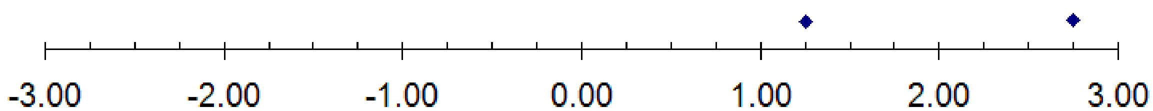
Alternate Solution: $\frac{3.5}{7} = \frac{1}{2}$, so $\frac{3}{7}$ is clearly less than that.

Example: Compare $\frac{7}{26}$ and $\frac{5}{14}$.

Solution: Since $7 \cdot 14 < 5 \cdot 26$, it follows that $\frac{7}{26} < \frac{5}{14}$.

Example: Graph each rational number as a point on the number line: $1\frac{1}{4}$, $2\frac{3}{4}$

Solution:



7.3 Exercises:

Prove that each number below is rational by expressing it as the ratio of two integers in the form $\frac{a}{b}$.

1. 3 **2.** .32 **3.** -1.471

4. $8.\overline{2}$ **5.** $-2.\overline{12}$ **6.** $4.2\overline{34}$

Convert to a decimal.

7. $\frac{35}{45}$ **8.** $\frac{12}{25}$ **9.** $\frac{18}{27}$

Place the following rational numbers on the same number line.

10. $-2\frac{1}{3}$ **11.** $2\frac{1}{8}$ **12.** $-\frac{7}{5}$

13. $\frac{3}{4}$ **14.** $\frac{2}{-5}$ **15.** $\frac{-5}{-8}$

Compare each pair of fractions using <, >, or = .

16. $\frac{5}{12}$ and $\frac{5}{13}$

17. $\frac{-5}{8}$ and $\frac{-8}{14}$

18. $\frac{5}{10}$ and $\frac{8}{18}$

19. $\frac{25}{26}$ and $\frac{19}{20}$

20. $\frac{-7}{22}$ and $\frac{-1}{3}$

21. List the numbers below from least to greatest:

a) $\frac{1}{4}, \frac{7}{6}, \frac{7}{24}, \frac{6}{7}, \frac{11}{20}, \frac{6}{13}, \frac{1}{3}$

b) $\frac{7}{13}, \frac{1}{4}, \frac{-1}{5}, \frac{5}{22}, \frac{11}{23}, \frac{-1}{4}, \frac{3}{11}$

c) $0.2, 0.23, 0.32, 0.\overline{3}, 0.\overline{23}, \frac{2}{9}, 0.\overline{32}, 0.231$

22. *Simplify: $\frac{(n-1)!}{n^{n-1}} \cdot n^n$

23. **Mr. and Mrs. Smith went to a party with three other married couples. Several people shook hands with some other people. No one shook hands with himself or herself and no one shook hands with his or her spouse. No one shook hands with the same person twice. After all this handshaking took place, Mr. Smith asked each person, “How many handshakes did you do?” Each person gave him a *different* answer. How many handshakes did Mrs. Smith do?

Assignment #7.4: Simplifying Expressions; Solving Equations by Adding and Subtracting Fractions

Example: $\frac{10}{3} - \frac{1}{3}$

Solution:
$$\frac{10}{3} - \frac{1}{3} = \frac{9}{3}$$
$$= 3$$

Example: $2\frac{1}{2} - \frac{1}{3}$

Solution:
$$2\frac{1}{2} - \frac{1}{3} = \frac{5}{2} - \frac{1}{3}$$
$$= \frac{5 \cdot 3}{2 \cdot 3} - \frac{1 \cdot 2}{3 \cdot 2}$$

$$= \frac{15}{6} - \frac{2}{6}$$

$$= \frac{13}{6} \text{ or } 2\frac{1}{6}$$

Example: $\dfrac{5}{6x} - \dfrac{1}{2x}$

Solution: $\dfrac{5}{6x} - \dfrac{1}{2x} = \dfrac{5}{6x} - \dfrac{1 \cdot 3}{2x \cdot 3}$

$$= \frac{5}{6x} - \frac{3}{6x}$$

$$= \frac{2}{6x}$$

$$= \frac{1}{3x}$$

To solve equations with fractions, we use the same techniques with fractions that we used previously with integers.

Example: Solve $x + \dfrac{5}{6} = \dfrac{5}{12}$

Solution: Subtract $\dfrac{5}{6}$ from each side to yield $x = \dfrac{-5}{12}$.

7.4 Exercises:

Simplify the following expressions.

1. $\dfrac{3a}{7} - \dfrac{3a}{14}$

2. $(\dfrac{1}{5} - \dfrac{2}{3}) - (\dfrac{6}{5} + \dfrac{1}{4})$

3. $-[\dfrac{1}{3} - \dfrac{1}{4}] - \dfrac{1}{5}$

4. $|1 - \dfrac{9}{7}| - |\dfrac{2}{3} - \dfrac{7}{6}|$

5. $\dfrac{4}{3}y + \dfrac{1}{6} + \dfrac{2}{3}y + \dfrac{1}{5}$

6. $\dfrac{a}{2} - \dfrac{1}{3} + \dfrac{a}{7} + \dfrac{1}{5}$

7. $\dfrac{a}{c} + \dfrac{b}{c}$

8. $\dfrac{5a}{2c} + \dfrac{7a}{6c}$

9. $\dfrac{a}{q} + \dfrac{t}{d}$

10. $\dfrac{a}{q} - \dfrac{t}{d}$

Solve the following equations.

11. $x + \frac{2}{5} = \frac{1}{2}$

12. $x + \frac{17}{20} = \frac{1}{40}$

13. $\frac{1}{20} = \frac{13}{40} + x$

14. $x - \frac{3}{5} = \frac{14}{45}$

15. $x + \frac{7}{15} = \frac{1}{6}$

16. $x + \frac{29}{15} = \frac{2}{10}$

17. $x - \frac{2}{45} = \frac{-4}{5}$

18. $x + \frac{11}{3} = \frac{11}{12}$

19. *The classic zoo problem:

Abe: "How many birds and how many beasts do you have in your zoo?"
Zookeeper: "We have 30 heads and 100 feet."
Abe: "I can't tell from that."
Zookeeper: "Oh, yes you can."
Can you tell how many birds were in the zoo?

Assignment #7.5: Simplifying Expressions with Multiplication and Division

To multiply rational numbers, change mixed fractions to improper fractions if necessary. Then multiply the numerators and multiply the denominators. Then reduce. Refreshingly, you do *not* need a common denominator.

Example: Multiply $2\frac{2}{5} \cdot \frac{5}{7}$. Note: this is the same as $\frac{12}{5} \cdot \frac{5}{7}$

Solution: Multiply the numerators and multiply the denominators. The result is $\frac{60}{35}$, which has a common factor of 5 in the top and bottom. So we reduce to $\frac{12}{7}$, or $1\frac{5}{7}$

Better Solution: In the original problem, $\frac{12}{5} \cdot \frac{5}{7}$, we see a common factor of 5 in the top and bottom. So we divide away the 5 at the start. This is sometimes called "canceling", and it looks like this: $\frac{12}{\cancel{5}} \cdot \frac{\cancel{5}}{7}$. The result is $\frac{12}{7}$.

To divide rational numbers, change mixed fractions to improper fractions if necessary. Then *invert* the *right* hand fraction, and then multiply the numerators and multiply the denominators. Then reduce.

Example: Divide $2\frac{2}{5} \div \frac{3}{5}$ Note: this is the same as $\frac{12}{5} \div \frac{3}{5}$

Solution: Invert the right hand fraction and multiply. Now the problem looks like this:

$\frac{12}{5} \cdot \frac{5}{3}$ Now we proceed as we would with multiplication. $\frac{12}{\cancel{5}} \cdot \frac{\cancel{5}}{3}$ reduces to $\frac{12}{3}$ or 4.

Why does the invert-and-multiply technique work? Let's first agree on another property of rational numbers.

Inverse property: For every rational number $\frac{a}{b}$ (with a and b *not* being zero), there exists an multiplicative inverse (called the reciprocal, or $\frac{b}{a}$), and the product of these two is 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

We now turn to the question of division being done with the invert-and-multiply technique.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} \text{ (multiplying the top and bottom by } \frac{d}{c}\text{)}$$

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} \text{ (using the inverse property on the bottom fractions)}$$

$$= \frac{a}{b} \cdot \frac{d}{c} \text{ (because dividing by 1 leaves the top the same)}$$

So $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, which is the invert-and-multiply technique.

7.5 Exercises:

Multiply or divide as appropriate. You may have to convert to improper fractions first. Be sure to reduce your answers to lowest terms. (For the sake of conceptual practice, leading to algebra, do *not* use your calculator.)

1. $\frac{3}{50}\cdot\frac{14}{9}$

2. $\frac{7}{35}\div\frac{11}{14}$

3. $\frac{-7}{20x}\div\frac{1}{4x}$

4. $\frac{3x}{16}\cdot\frac{8x}{9x}$

5. $1\frac{1}{9}\div\frac{5}{6}$

6. $\frac{18}{60y}\cdot\frac{5y}{6}$

7. $\frac{5}{3}\cdot\frac{2a}{7}\cdot\frac{6}{10a}$

8. $\frac{-x}{20}\cdot\frac{5}{4x}$

9. $\frac{1}{18y}\div\frac{1}{54y}$

10. $\frac{2}{3x}\cdot\frac{-9}{5}\cdot\frac{20x}{8}$

11. $200\cdot\frac{3}{2}\cdot\frac{-1}{7}$

12. $1\frac{5}{8}\mathrm{c}\cdot\frac{16}{26c}$

13. $\frac{-4}{3x}\div\frac{1}{48y}$

14. $\frac{-a}{c}\div\frac{b}{c}\cdot\frac{b}{a}$

15. $\frac{a}{b}\cdot\frac{b}{a^2}$

16. $3(\frac{2}{3}x-\frac{1}{6})$

17. $\frac{-1}{3}(2x+6)$

18. $4(\frac{-1}{3}x+\frac{3}{2})$

19. $\frac{2}{3}(\frac{7}{4}x-\frac{6}{5})+3(\frac{x}{36}+\frac{1}{5})$

20. $\frac{1}{5}(10y+2)-(-y-\frac{1}{4})$

21. $\dfrac{\frac{6}{14}}{\frac{6}{7}}$

22. $\dfrac{\frac{26}{a}}{\frac{52}{12a}}$

23. *Daniel and Carly observed that a microscopic slide had 7500 bacteria on it, which were dying at the rate of 150 per hour. At the same instant, they noted that another slide had 4500 bacteria, which were increasing at the rate of 50 per hour. How long will it be until the bacteria count on both slides will be the same?

Assignment #7.6: Solving Equations by Multiplying and Dividing Fractions

To solve equations with fractions, we use the same techniques with fractions that we used previously with integers.

Example: Solve $\frac{-4}{3}\mathrm{a}=\frac{2}{9}$

Solution: $\frac{3}{-4}\cdot\frac{-4}{3}\mathrm{a} = \frac{2}{9}\cdot\frac{3}{-4}$

$$a = \frac{-1}{6}$$

Example: Solve $\frac{2}{3}x + \frac{5}{6} = \frac{5}{12}$

Solution: Subtract $\frac{5}{6}$ from each side to yield $\frac{2}{3}x = \frac{-5}{12}$

Now eliminate the $\frac{2}{3}$ by multiplying both sides by its reciprocal $\frac{3}{2}$

$$\frac{3}{2} \cdot \frac{2}{3}x = \frac{-5}{12} \cdot \frac{3}{2}$$

$$1x = \frac{-15}{24}$$

$$x = -\frac{5}{8}$$

Example: Solve $\frac{x}{3} + \frac{5}{6} = \frac{5}{12}$

Solution: Subtract $\frac{5}{6}$ from each side to yield $\frac{x}{3} = \frac{-5}{12}$

Now eliminate the denominator of $\frac{x}{3}$ by multiplying both sides by 3 (also known as $\frac{3}{1}$)

$$\frac{3}{1} \frac{x}{3} = \frac{-5}{12} \cdot \frac{3}{1}$$

$$1x = \frac{-15}{12}$$

$$x = -\frac{5}{4}$$

Example: Solve $5x + \frac{5}{6} = \frac{5}{12}$

Solution: Subtract $\frac{5}{6}$ from each side to yield $5x = \frac{-5}{12}$

Now eliminate the 5 from 5x by dividing both sides by 5.

$$x = \frac{-5}{12} \div \frac{5}{1}$$

$$x = \frac{-5}{12} \cdot \frac{1}{5} \qquad \text{(Note the inversion!)}$$

$$x = -\frac{1}{12}$$

7.6 Exercises:

Solve for x.

1. $3x = -7$

2. $4x = \frac{1}{3}$

3. $-3t = \frac{6}{7}$

4. $\frac{2}{5}y = -12$

5. $\frac{-11}{4}p = -121$

6. $\frac{7}{8}z = \frac{-21}{20}$

7. $\frac{-y}{7} = \frac{-3}{14}$

8. $-\frac{124}{5}w = \frac{31}{15}$

9. $-\frac{2}{5}x + \frac{1}{3} = \frac{11}{12}$

10. $5x - \frac{31}{4} = -\frac{7}{8}$

11. $\frac{3}{4}x + \frac{5}{48} = \frac{5}{12}$

12. $6x - \frac{9}{4} = \frac{3}{8}$

13. $\frac{2}{3}x + 2\frac{1}{3} = \frac{13}{6}$

14. $-7x - \frac{3}{18} = \frac{5}{6}$

15. $\frac{3x}{4} + \frac{1}{9} = \frac{1}{4}$

16. $\frac{1}{20} = \frac{17}{40} + 3x$

17. *One-third of all the seventh grade students gave their reports on Monday, one-fourth of the students gave their reports on Tuesday, one-fifth of the group gave theirs on Wednesday, and three students gave their reports on Thursday. If the remaining one-sixth gave their reports on Friday, how many students were in the seventh grade?

Assignment #7.7: Solving Equations by Cross-Multiplying

When two fractions are equal, their **cross-products** are equal. For example, $\frac{1}{8} = \frac{3}{24}$ and their cross products are also equal: $1 \cdot 24 = 8 \cdot 3$.

<u>Example</u>: Solve for the variable in $\frac{5}{8} = \frac{15}{x}$

<u>Solution</u>: Cross multiply to get $5x = (8)(15)$.

$$x = \frac{8 \cdot 15}{5} = 8 \cdot 3 = 24$$

Example: Solve $\frac{5}{8-x} = \frac{7}{x+4}$

Solution: Cross multiply to get $5(x + 4) = (7)(8 - x)$.

$$\begin{aligned} 5x + 20 &= 56 - 7x \\ \underline{+\ 7x} \quad & \quad \underline{+\ 7x} \\ 12x + 20 &= 56 \\ \underline{-20} \quad & \quad \underline{-20} \\ 12x &= 36 \\ x &= 3 \end{aligned}$$

7.7 Exercises:

Solve.

1. $\frac{5}{8} = \frac{15}{x}$ **2.** $\frac{5}{8} = \frac{x}{2}$ **3.** $\frac{5x}{8} = \frac{-3}{24}$

4. $\frac{x}{8} = \frac{7}{14}$ **5.** $\frac{42}{3} = \frac{7}{2x}$ **6.** $\frac{-5}{8} = \frac{10}{x+1}$

7. $\frac{5}{3} = \frac{20}{x+3}$ **8.** $\frac{3}{x-2} = \frac{12}{24}$ **9.** $\frac{5}{8} = \frac{5x}{8x}$

10. $\frac{3}{51} = \frac{x}{x+16}$ **11.** $\frac{7m+3}{2m+2} = 4$ **12.** $\frac{x+2}{x+3} = \frac{-3}{8}$

13. $\frac{2}{x-3} = \frac{4}{2x+1}$ **14.** $\frac{4}{x+2} = \frac{1}{x-7}$ **15.** $\frac{4x-2}{4} = \frac{2x-1}{2}$

16. What integer can be added to both the numerator and denominator of the fraction 3/5 so that the resulting fraction will be equivalent to 8/9 ?

17. On Charlie Brown's first two tests, he scored 78% and 83%. What must he score on the third test to have an <u>average</u> for the three tests of 84%?

18. *In the Htam family, each daughter has the same number of brothers as she has sisters, and each son has twice as many sisters as he has brothers. How many sons and how many daughters are in the Htam family?

Assignment #7.8: Solving Equations with Decimals

7.8 Exercises:

Solve and check.

1. $\frac{5}{x-4} = \frac{10}{x}$ **2.** $\frac{x+2}{4} = \frac{x+2}{10}$ **3.** $\frac{8x-1}{6} = \frac{4x+5}{3}$

Solve and check. (The following problems should be done **without** a calculator so that you can practice working with decimals.)

4. $0.1p + 5.66 = 6.48$

5. $-4x - 6.24 = 11.76$

6. $0.7y + 9.41 = 12.35$

7. $3p + 5.13 = 27.93$

8. $2x - (-4.43) = 7.03$

9. $-7y + (-0.16) = 51.64$

10. $2m - (-0.48) = 14.88$

11. $3(3x + 2) = 4.5(2x + 4) - x$

These problems may be done **with** a calculator…the numbers are not quite as "nice".

12. $-3.8m - (-4.37) = 36.67$

13. $4.7m - 9.39 = -44.64$

14. $6.3\,p + (-3.41) = -38.69$

15. On Linus' first two tests, he scored 88% and 73%. What must he score on the third test to have an average for the three tests of 87%?

16. *Angel has $15.25 worth of nickels and dimes in her piggy bank. If she has an odd number of dimes and if she has more dimes than nickels, what is the largest percentage of her coins that could be nickels?

Assignment #7.9: Using the Multiplication Property of Equality to Solve Equations with Fractions and Decimals

The multiplication property of equality is a powerful tool that can be used to eliminate fractions and decimals in equations so that they become easier to solve.

Example: Solve $\frac{1}{3}m + \frac{3}{2}m = 3m + \frac{3}{2} + \frac{5}{6}m$.

Solution: Calculate the least common multiple of the fractions in the equation; in this case it is 6. Multiply every term of the equation, on both sides, including the non-fraction terms, by 6.

$$6 \cdot \frac{1}{3}m + 6 \cdot \frac{3}{2}m = 6 \cdot 3m + 6 \cdot \frac{3}{2} + 6 \cdot \frac{5}{6}m$$

$$2m + 9m = 18m + 9 + 5m$$

$$11m = 23m + 9$$

$$-11m \quad -11m$$

$$0 = 12m + 9$$

$$-9 \qquad -9$$

$$-9 = 12m$$

$$\frac{-3}{4} = m$$

Example: Solve $0.64y - 3.2y + 4.8 = 1.24y - 2.8$

Solution: Determine the power of 10 that will eliminate the decimal sign from the term with the largest number of digits to the right of the decimal point. In this example that is two digits, found in 0.64y and 1.24y, so we multiply each side of the equation by 100.

$$
\begin{aligned}
100 \cdot .64y - 100 \cdot 3.2y + 100 \cdot 4.8 &= 100 \cdot 1.24y - 100 \cdot 2.8 \\
64y \quad - \quad 320y \quad + \quad 480 &= 124y - 280 \\
-256y \quad + \quad 480 &= 124y - 280 \\
+256y \qquad\qquad &\quad +256y \\
480 &= 380y - 280 \\
+280 \qquad &\quad +280 \\
760 &= 380y \\
2 &= y
\end{aligned}
$$

7.9 Exercises:

Solve and check. (The following problems should be done **without** a calculator so that you can practice working with fractions and decimals.) Use the multiplication property of equality to eliminate the fractions and decimals in these equations.

1. $\frac{7}{2}x + \frac{1}{2}x = 3x + \frac{3}{2} + \frac{5}{2}x$

2. $\frac{2}{3} + \frac{1}{4}p = \frac{1}{3}$

3. $\frac{2}{3} + 3t = 5t - \frac{2}{15}$

4. $1.1p - 6 = 0.4p + 5$

5. $\frac{2}{7}f + \frac{1}{2}f = \frac{3}{4}f + 1$

6. $-0.\overline{5}x + \frac{4}{9} = \frac{2}{3}$

7. $\frac{5}{3} + \frac{2}{3}x = \frac{25}{12} + \frac{5}{4}x + \frac{3}{2}$

8. $3.\overline{2}x - \frac{1}{3} = \frac{2}{9}$

9. $\frac{4}{5}y - \frac{3}{4}y = \frac{3}{10}y - 1$

These problems may be done **with** a calculator…

10. $2.1y + 45.2 = 3.2 - 8.4y$

11. $1.03 - 0.62a = 0.71 - 0.22a$

12. $0.42 - 0.03m = 3.33 - m$

13. $4.7m - 9.39 = -44.64$

14. $6.3p + (-3.41) = -38.69$

15. *Find as many whole number values for X and Y as you can such that $\frac{X}{12} = \frac{28}{Y}$.

Assignment #7.10: Evaluating Formulas, Solving for a Variable in a Formula

A **formula** is a mathematical expression using two or more variables. For example, the formula $A = L \cdot W$ relates the length (L) of a rectangle and its width (W) to its area (A).

Example: Evaluate $A = LW$ for $L=3$ and $W=7$. Then identify the formula by its use.

Solution:
$$A = LW$$
$$A = 3 \cdot 7$$
$$A = 21$$

$A = LW$ is the formula to find the area (A) of a rectangle, given its length (L) and width (W).

Sometimes it is useful to solve for one variable in terms of the others.

Example: If $3x - 4y = 12$, express y in terms of x.

Solution:
$$\begin{aligned} 3x - 4y &= 12 \\ \underline{-3x} \quad & \quad \underline{-3x} \\ -4y &= -3x + 12 \\ y &= \frac{-1}{4}(-3x + 12) \\ y &= \frac{3}{4}x - 3 \end{aligned}$$

7.10 Exercises:

Evaluate each formula for the given values of the variables. Then, if possible, identify the formula.

1. $V = LWH$ for $L = 2$, $W = 3$, $H = 4$

2. $V = Bh$ for $B = 20\text{cm}^2$, $h = 17\text{cm}$

3. $E = MC^2$ for $M = 7$, $C = 3$

4. $C = 2\pi R$ for $\pi = \frac{22}{7}$, $R = 14\text{in}$

5. $A = \pi r^2$ for $\pi = \frac{22}{7}$, $r = 14\text{cm}$

6. $A = 4\pi R^2$ for $\pi = \frac{22}{7}$, $R = 21$

7. $y = \frac{4}{3}x - 6$ for $x = -9$

8. $V = \frac{\pi r^2 h}{3}$ for $r = 3$, $h = 6$

9. $C = \frac{5}{9}(F - 32)$ for $F = 99$

10. $y = \frac{-2}{5}x + 12$ for $x = 4$

Solve for the indicated variable:

11. $C = 2\pi R$; for R

12. $V = \frac{\pi r^2 h}{3}$; for h

13. $A = P + Prt$; for t

14. $P = 2L + 2W$; for L

15. $4x + 5y = 20$; for y

16. $y = \frac{2}{3}x - 7$; for x

17. $C = \frac{5}{9}(F - 32)$; for F

18. $y = \frac{-2}{5}x + 12$; for x

Chapter 8: Ratios and Proportions

A **ratio** is a comparison of two numbers using division. Thus the ratio of 2 to 3 can be expressed as $\frac{2}{3}$. It can also be expressed with the slanting fraction bar as $^2/_3$ or with the colon notation as 2 : 3, or simply in English as "2 to 3".

We will typically express a ratio as a fraction in lowest terms, and this is what will be meant in this book unless otherwise specified. However, the colon notation is still handy, especially if we want to compare more than two numbers. If there are 24 violins, 10 cellos, and 6 violas in the string ensemble, then the ratio of violins to cellos to violas can be compactly expressed as 24:10:6, which we should reduce to 12:5:3.

One often sees the colon notation in recipes, where there are several ingredients and the total <u>amount</u> of ingredients is not what matters. Rather, it is the <u>ratio</u> of the ingredients that matters. A salad dressing might be specified as "3 parts vinegar, 2 parts oil, and 1 part lemon juice", which means a ratio of 3:2:1.

Assignment #8.1: Ratios

8.1 Exercises:

For problems 1 through 8, write each ratio as a fraction *in lowest terms*.

1. $\frac{24}{16}$ **2.** $\frac{8}{64}$ **3.** 16:36

4. 18 out of 36 **5.** 24 out of 60 **6.** 48:144

7. 60 to 200 **8.** 9 to 24 **9.**

10. A basketball team won 12 games and lost 6. What is its ratio of wins to losses?

11. A basketball team won 12 games and lost 6. What is its ratio of wins to total games?

12. A popular band needed 250 hours of recording studio time to produce a hit CD which contained 45 minutes of music. What is the ratio of minutes of CD music to minutes of work needed to record it?

13. Six buses took 425 students to the state capital in Olympia. Four adults were on each bus. What was the ratio of adults to students?

14. In a group of people, the ratio of men to children is 3:5 and the ratio of women to children is 5:8. Are there more women or men in the group?

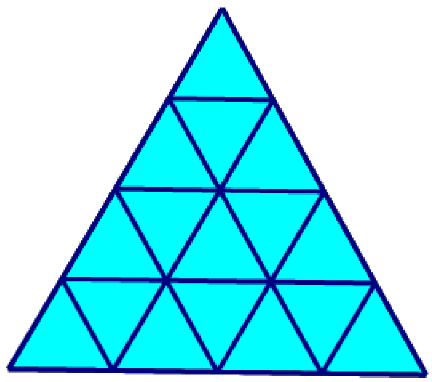
15. A plane with a capacity of 225 passengers had 12 passengers in first class and 113 passengers in coach. What was the ratio of full passenger seats to empty passenger seats?

16. A builder used 12 parts sand, 15 parts gravel, 6 parts cement, and 3 parts water in a concrete mix. Write this mixture as a ratio in lowest terms.

17. The Smithereens traveled 319.8 miles and used 16.4 gallons of gas in the process. What was the ratio of miles to gallons? Express this as a rate of fuel consumption (mileage) in miles per gallon.

18. Ben bought 4 ½ pounds of bananas for \$3.96. What was the unit price of the bananas in dollars per pound?

19. *How many triangles are in the figure below?



Assignment #8.2: Proportions

A **proportion** is an equation that claims that two ratios are equal.

<u>Example</u>: $\frac{4}{6} = \frac{6}{9}$ is a proportion.

As we discussed in the last chapter, if a proportion is true (that is, if the two ratios are equal), the **cross-products** are also equal.

<u>Example</u>: $\frac{4}{6} = \frac{6}{9}$ is true, and thus the cross products $4 \cdot 9$ and $6 \cdot 6$ are equal (both are 36).

$$\frac{4}{6} = \frac{6}{9}$$

8.2 Exercises:

For problems 1 through 8, tell whether the statements about the ratios are true or false. Show your work.

1. $\frac{5}{9} = \frac{15}{27}$

2. $5:3 = 15:9$

3. $\frac{3}{5} < \frac{5}{7}$

4. $4:9 < 5:11$

5. $\frac{3}{39} > \frac{4}{52}$

6. $76:100 > 9:12$

7. $3:5 < 55:100$

8. Four out of 28 is the same thing as 5 out of 35.

9. One basketball player made 5 out of 16 shots while another made 3 out of 10. Which one had the more success in shooting the ball?

10. In a certain bakery, a 12-ounce loaf of bread sells for $2.00 and a 16-ounce loaf sells for $2.50. Which loaf is the better buy?

Solve each proportion below.

11. $3:2 = 15:N$

12. $3:N = 75:100$

13. $\frac{N}{8} = \frac{24}{32}$

14. $\frac{25}{N} = \frac{15}{9}$

15. $\frac{7}{16} = \frac{X}{4.8}$

16. (c) $\frac{3.5}{6.2} = \frac{7.35}{B}$

17. $\frac{5}{4} = \frac{X+5}{16}$

18. $\frac{15}{M-3} = \frac{3}{14}$

For each problem below, first ***write and label a proportion***. Each proportion should include the units for each quantity (an abbreviation is fine). Then solve each proportion to answer the question.

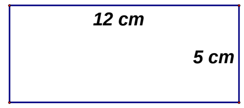
19. At a picnic, the ratio of adults to children is 3 to 5. If there are 21 adults, how many children are there?

20. If each group of three scouts can eat ten Krispy-Kreme doughnuts, how many doughnuts are needed to satisfy 24 scouts?

21. In a recent basketball game, 2 out of 5 shots were successful. Eighty shots were attempted. How many shots were successful?

22. In a recent basketball game, 7 shots were made for every 5 that missed. If 21 were made, how many shots were taken in total (both made and missed)?

23. *A rectangle is 12 cm long and 5 cm wide. Find the dimensions of another rectangle such that the ratio of the perimeter of the second rectangle to the perimeter of the first is 2:1 and the ratio of the area of the second to the area of the first is also 2:1.



Assignment #8.3: Proportions II

8.3 Exercises:

For problems 1 through 5, tell whether the proportions are ***TRUE*** or ***FALSE***. Show your work.

1. $\frac{4}{7} = \frac{32}{56}$

2. $\frac{9}{4} = \frac{63}{32}$

3. $\frac{\frac{1}{3}}{7} = \frac{1\frac{1}{3}}{28}$

4. $\frac{1.25}{7} = \frac{7\frac{1}{2}}{42}$

5. $\frac{3}{2\frac{1}{4}} = \frac{12}{9}$

Solve:

6. $\frac{39}{54} = \frac{13}{N}$

7. $\frac{24}{N} = \frac{4}{3}$

8. $\frac{7}{12} = \frac{4}{N}$

9. $\frac{N}{2/3} = \frac{3/5}{10}$

Write and solve proportions to answer the following questions.

10. The ratio of raisins to oats is 1:8 in a bowl of oatmeal. If there are 18 ounces of oatmeal, how many ounces of raisins and how many ounces of oats are in the bowl?

11. The ratio of Andy's snake's weight to Destiny's frog's weight is 5:6. Andy's snake weighs 85 lbs. How much does Destiny's frog weigh?

12. How many girls are in a class of 32 students if the ratio of girls to boys is 3:5?

13. Jason has too much time on his hands, so he figures out that his favorite snack mix contains peanuts and cashews in a ratio of 5:4. He decides to make some mix, using 520 grams of peanuts. How many grams of cashews should he use?

14. The ratio of full seats to the total number of seats in the passenger section of Mr. Milloy's flight is 3 to 7. There are 28 empty seats. How many passengers are on the plane?

15. A school system has 9000 students and a teacher to pupil ratio of 1:30. How many more or fewer teachers would it take to reduce this ratio to 1:25?

16. At Mister Softee the employment office accepts nine applicants for every two that are rejected. If 1408 high school students applied for jobs last summer, how many were accepted?

17. At Mister Meanie, the employment office <u>rejects</u> seven applicants for every eleven that apply. If they hired 24 high school students, how many applied?

18. A recipe calls for 4 ½ cups of flour for 72 cookies. How many cups of flour would be needed for 160 cookies?

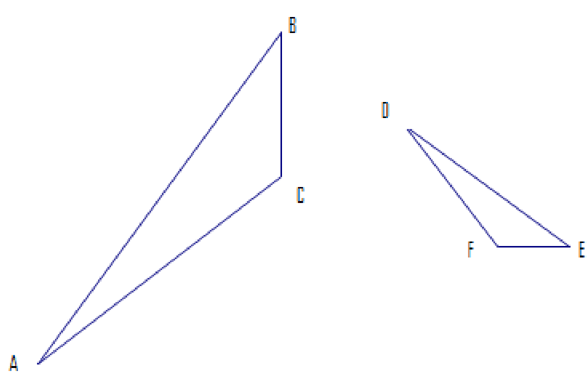
19. The Preussen was the largest sailing ship ever built. Built in Germany in 1902, it had 5 masts and could carry 7300 metric tons of cargo. The Preussen was 433 feet long and 54 feet wide. If a model of the ship is to be 26 inches long, how wide should the model be (to the nearest hundredth of an inch)?

20. *X and Y are positive integers. The ratio of their difference to their sum is 1:7. The ratio of their sum to their product is 7:24. Find X and Y.

Assignment #8.4: Similar Geometric Figures

Geometric figures with the same shape (though not necessarily the same size) are said to be **similar**. The ratios of the lengths of corresponding sides in such figures are always equal.

Triangles that have the same angles are always similar. As we noted above, the corresponding sides of similar triangles are in the same ratio. This ratio is called the scale factor. If triangle ABC is similar to triangle DEF, we write this as $\Delta ABC \sim \Delta DEF$. Then, for example:



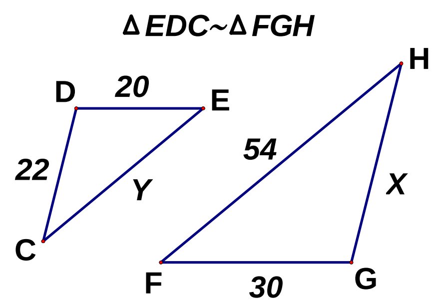
$$\frac{AB}{DE} = \frac{BC}{EF}$$

and

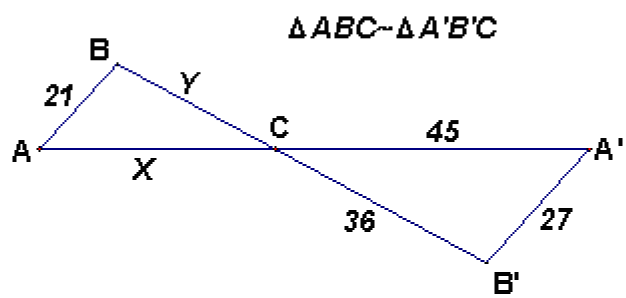
$$\frac{AB}{BC} = \frac{DE}{EF}$$

8.4 Exercises:

1. Find the length of side X in the similar triangles below.
2. Find the length of side Y in the similar triangles below.

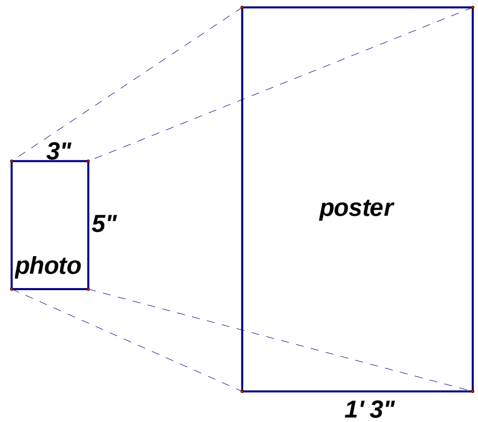


3. Find the length of side X in the similar triangles below.
4. Find the length of side Y in the similar triangles below.

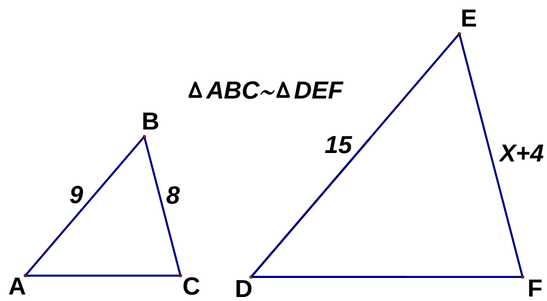


5. A person 2 meters tall casts a shadow that is 6 meters long. At the same time, a nearby flagpole casts a shadow that is 54 meters long. How high is the flagpole?
6. A ladder was leaning against a house. McLean took 3 steps up the ladder. Her feet were then 2 feet above the ground. She took 4 more steps up the ladder. At that point, how high were her feet off the ground?
7. If 8x = 7y, what is the ratio of x to y?
8. Lindsey is standing nine feet from a street lamp pole. The light from the lamp gives Lindsey a shadow that is 15 feet long. If she is 5 feet tall, how tall is the lamp post?
9. Lindsey's little brother is standing an unknown distance from the same lamppost. If he is 4 feet tall and the total distance from the base of the lamp post to the tip of his shadow is 20 feet, how far is he from the post?
10. The three angles of a triangle are in the ratio of 3:2:1. What are the three angles?

11. If the poster is an enlargement of the photo, what is the height of the poster?



12. $\Delta ABC \sim \Delta DEF$. Find the value of X.



13. The ratio of boys to girls in a group of students was 3 to 5. Then 24 girls left the group and 24 more boys joined it. The ratio of boys to girls became 5 to 3. How many boys and girls were in the original group?

14. Solve for w, x, y, and z in the following equation chain. All of the ratios are equal.

$$\frac{4}{7} = \frac{16}{w} = \frac{20}{35} = \frac{x}{21} = \frac{140}{y} = \frac{z}{140}$$

15. *Max sends two toy ducks on a collision course. One is traveling at 50 miles/hr and the other at 70 miles/hr. How far apart will these ducks be one minute before they collide?

Assignment #8.5: Dimensional Analysis

Consider the following examples:

55 miles per hour
16 students per class
3 inches per day

Each of these quantities is an example of a rate. Rates frequently can be recognized because they contain the word "per" or "for each."

Example: Suppose that apples cost \$1.69 per pound. How much does it cost to purchase 5 pounds of apples?

Solution: (5 pounds of apples) $\frac{\$1.69}{pound} = \$8.45$.

A conversion factor is a special kind of rate. These are the rates that come from knowing that there are 5280 feet in a mile or that there are 60 minutes in an hour, or that there are 2 cups in a pint. The associated rates are called conversion factors or unit multipliers.

$$\frac{5280\,feet}{1mile};\ \frac{60\min}{1hour};\ \frac{2cups}{1p\,\text{int}}$$

A conversion factor is always equal to 1. Since any fraction multiplied by one remains unchanged, we can use unit multipliers, or conversion factors, to transform units without changing the value of a fraction.

Example: If a cheetah, the fastest known land animal, can achieve a speed of 70 miles per hour, how fast is this in feet per second?

Solution: $\frac{70miles}{hour}\cdot\frac{hour}{60\min}\cdot\frac{1\min}{60\sec}\cdot\frac{5280\,feet}{mile}\approx 103\,feet/\sec$

Example: If a cheetah runs 70 miles per hour and could continue at this pace for 1 day, how far could it travel?

Solution: 1 day$\cdot\frac{24hours}{1day}\cdot\frac{70miles}{1hour}$ = 1680 miles

8.5 Exercises:

1. If each of the four sections of Math 7 has 12 students, each student took one test, and each test contained 13 questions, how many problems do the Math 7 teachers have to look at tonight?

2. Aidan can type 30 words per minute. At this rate, how many hours will it take him to type his 5000 word history essay?

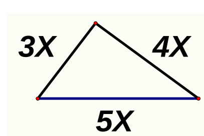
There are 20 koops in a meep and 50 meeps in a vupp.

3. How many koops are there in 15 vupps?

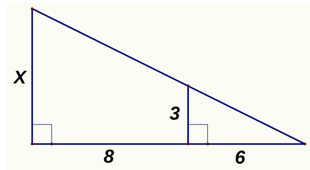
4. How many vupps are there in 10,000 koops?

5. Isabella's favorite food is fig newtons. She eats five fig newtons every day. The cookies are sold in packages of 24, and one package costs \$3.99. How much has Isabella spent on fig newtons over the past two years?

6. The lengths of the three sides of a triangle are in a ratio of 5:4:3. The perimeter of the triangle is 204 centimeters. What are the lengths of the three sides of the triangle?



7. Find the length of the side X in the figure shown.



8. Leaf-cutter ants that live in Central and South America weigh about 1.5 grams. One ant can carry a 4-gram piece of leaf about the size of a dime. If a person could carry proportionally as much as the leaf-cutter ant, how much could a 55-kilogram math student carry?

9. The full grown leaf-cutter is about 1.27cm long. The ants walk up to 0.4 km from home each day. If a person could walk a proportional distance, how far would a 1.65m tall math student walk?

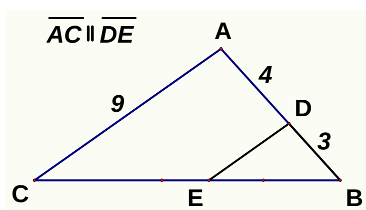
Assignment #8.6: More Practice with Proportions and Dimensional Analysis

8.6 Exercises:

1. $\dfrac{X+2}{3} = \dfrac{X+9}{10}$

2. $\dfrac{N+2}{N-2} = \dfrac{3}{5}$

3. Find the length of $\overline{DE}$ in the figure shown.



4. If the length of $\overline{CB}$ in the figure shown is 12, what is the length of $\overline{EB}$?

5. A quarter has the same weight as 2 pennies. If a pound of quarters is worth $25, then how much is a pound of pennies worth?

6. The ratio of the number of objects in two sets is 7 to 5. If the <u>smaller</u> set has 280 objects, how many objects are in the larger set?

7. The ratio of the number of objects in two sets is 7 to 5. If the <u>larger</u> set has 280 objects, how many objects are in the smaller set?

8. Assembly line X produces 5 cars in the same time that assembly line Y produces 4 cars. Assembly line Z produces 3 cars in the same time that assembly line Y produces 2 cars. What is the ratio of cars produced by assembly line X to assembly line Z?

9. In a group of men, women and children, the ratio of men to children is 1:2 while the ratio of women to children is 1:3. What is the ratio (in lowest terms) that compares the total number of men and women to the number of children?

10. A cookie recipe that makes one dozen cookies calls for 2 teaspoons of cinnamon. You have thirty-six friends coming to a party and you expect each one to eat two cookies. How many Tablespoons of cinnamon will you need?

11. The Earth rotates once per day on its axis. Figure out the circumference of the Earth and determine the speed you are going around while standing still on the Equator. Give this speed in miles per hour.

12. Convert the speed you found in question #11 into the units of meters per second.

13. At the neighborhood bakery a baker's dozen has 13 items. If you have 52 people coming to your birthday party and you want to have 4 cookies per person, how many dozen cookies should you buy from the bakery?

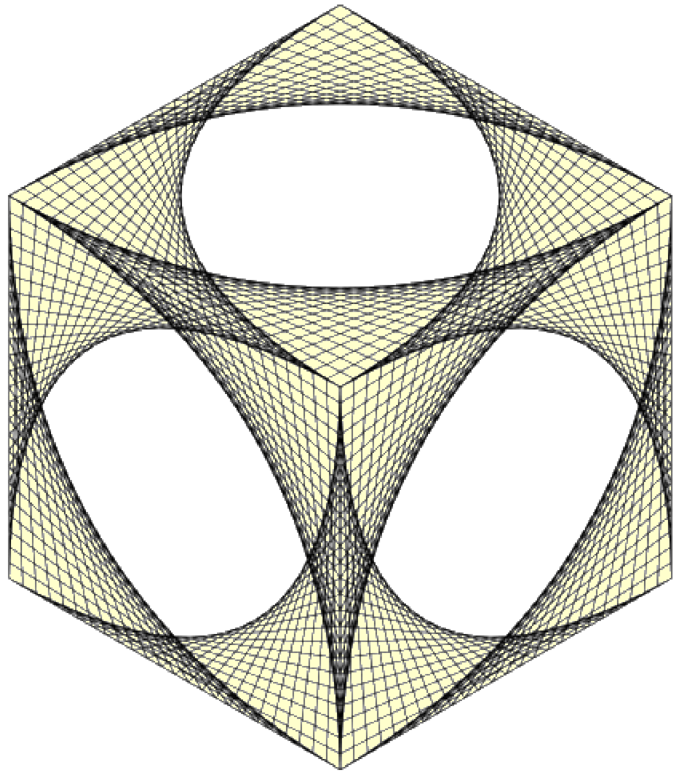
14. A stack of 100 nickels has a height of 6.25 inches. What is the value, in dollars, of an 8-foot stack of nickels? Express you answer to the nearest hundredth.

15. It takes 12 minutes to cut a log into 4 pieces. How long would it take to cut a similar log into 6 pieces?

16. You're throwing a pizza party for 15 and figure each person might eat 4 slices. How much is the pizza going to cost you? You call up the pizza place and learn that each pizza will cost you $14.78 and will be cut into 12 slices. You tell them you'll call back. Do you have enough money if you have $75?

17. *Divide 100 into four unequal whole numbers (whose sum is 100, of course) such that if 4 is subtracted from the first number, 4 is added to the second number, the third number is multiplied by 4 and the fourth number is divided by 4, an identical result is obtained in each instance. What are the four numbers?

MATH 7
PRE-ALGEBRA &
PROBLEM-SOLVING



BOOK III

LAKESIDE MIDDLE SCHOOL
Ms. Canino & Ms. O'Neill

MATH 7

PRE-ALGEBRA & PROBLEM-SOLVING

BOOK III

Lakeside School, Seattle

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Chapter 9: Exponents and Scientific Notation

Assignment #9.1: Exponent Rules

These are the formal rules of exponents with which you may or may not have some experience:

$x^a \cdot x^b = x^{a+b}$	$\dfrac{x^a}{x^b} = x^{a-b}$	
$(x^a)^b = x^{ab}$	$(xy)^a = x^a y^a$	
$x^0 = 1$		$(x \neq 0)$
$x^{-a} = 1/x^a$	$\dfrac{1}{x^{-a}} = x^a$	$(x \neq 0)$
$\left(\dfrac{x}{y}\right)^{-a} = \left(\dfrac{y}{x}\right)^a$		$(x \neq 0, y \neq 0)$

These rules come from making observations about general rational numbers expressed as variables:

To begin, let's derive the rules that $x^a \cdot x^b = x^{a+b}$ and $\dfrac{x^a}{x^b} = x^{a-b}$

$$\begin{aligned} x^a \cdot x^b &= x \cdot x \cdot x \cdot x \cdot x \text{ (a times)} \cdot x \cdot x \cdot x \cdot x \text{ (b times)} \\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \text{ (a+b times)} \\ &= x^{a+b} \end{aligned}$$

$$\begin{aligned} \frac{x^a}{x^b} &= \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \text{ (a times)}}{x \cdot x \cdot x \cdot x \text{ (b times)}} \\ &= x \cdot x \cdot x \text{ (a-b times)} \qquad \text{(after canceling)} \\ &= x^{a-b} \end{aligned}$$

Using the first rule,

$$\begin{aligned} (x^a)^b &= x^a \cdot x^a \cdot x^a \cdot \text{ [...b times]} \\ &= x^{a+a+a+\ldots\text{[...b times]}} \\ &= x^{ab} \end{aligned}$$

Using the first rule and the associative and commutative laws of multiplication,

$$(xy)^a = xy \cdot xy \cdot xy \cdot \text{ [...a times]}$$

$= x^a y^a$

From the second rule, we are able to show that $x^0 = 1$, since

$$x^{a-a} = x^0$$

$$\text{and } x^{a-a} = \frac{x^a}{x^a}$$

$$\text{so } x^0 = \frac{x^a}{x^a} = 1 \qquad \text{(if } x \neq 0\text{)}$$

Now using this and the second rule, we get that $x^{-a} = \frac{1}{x^a}$

$$\frac{1}{x^a} = \frac{x^0}{x^a} = x^{0-a}$$

$$= x^{-a} \qquad \text{(if } x \neq 0\text{)}$$

From this, we can say:

$$\frac{1}{x^{-a}} = \frac{1}{\frac{1}{x^a}} = x^a$$

And finally,

$$\left(\frac{x}{y}\right)^{-a} = \frac{1}{\left(\frac{x}{y}\right)^a} = \frac{1}{\frac{x^a}{y^a}} = \frac{1}{1} \div \frac{x^a}{y^a} = \frac{1}{1} \cdot \frac{y^a}{x^a} = \left(\frac{y}{x}\right)^a$$

Examples of simplifying or rewriting expressions by using the exponent rules:

a. $(-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64}$

b. $(3^{-2})^{-2} = 3^4 = 81$

c. $(3 \cdot 2)^{-3} = (6)^{-3} = \frac{1}{216}$

d. $2 \cdot 4^{-2} = 2 \cdot \frac{1}{16} = \frac{1}{8}$

e. $\frac{2^3 \cdot 2^{-2}}{2^5} = \frac{2^1}{2^5} = 2^{-4} = \frac{1}{16}$

f. $\frac{x^2 \cdot x^{-3}}{x^4 \cdot x^{-9}} = \frac{x^{-1}}{x^{-5}} = x^{-1-(-5)} = x^4$

g. $\frac{12x^3 - 8x^7}{4x^3} = \frac{12x^3}{4x^3} - \frac{8x^7}{4x^3} = 3 - 2x^4$

h. $(3^{-1}+4^{-1})^{-1} = (\frac{1}{3} + \frac{1}{4})^{-1} = (\frac{7}{12})^{-1} = \frac{12}{7}$

9.1 Exercises:

Use the rules of exponents to compute. Your answer should not have any exponents.

1. 3^{-1}

2. 2^{-3}

3. -6^{-2}

4. $(-3)^{-2}(2^{3})$

5. $5^{5} \cdot 5^{-3}$

6. $\left(\frac{3}{2}\right)^{-2}$

7. $3^{3} \cdot 2^{3}$

8. $7^{-2}(-2^{7})$

9. $2^{-6} \cdot (2^{4})^{2}$

10. $\left(\frac{6^2}{6^3}\right)^{-1}$

11. $(4^{2})^{3} \div (-4^{2})$

12. $(-5)^{-2}$

13. $(4^{2})^{-2}$

14. $(5 \cdot 3)^{-2}$

15. $3 \cdot 5^{-3}$

16. $\frac{3^{12} \cdot 3^{-7}}{3^{7}}$

17. $\left(3^{-2} + 5^{-1}\right)^{-1}$

Simplify. Your answers will usually (but not always) have exponents.

18. $x^{3} \cdot x^{12}$

19. $(x^{-1})^{-1}$

20. $x^{13} \cdot (x^{7})^{3}$

21. $\frac{x^{5} \cdot x^{-3}}{x^{6} \cdot x^{-7}}$

22. *If Tyler gives you $6, both you and Tyler will have the same amount of money. If, however, you give Tyler $6 instead, he will have twice as much as you. How many dollars does Tyler have?

Assignment #9.2: More Practice with Exponents

9.2 Exercises:

Compute. Do not use your calculator.

23. $(-3)^{-2}$

24. $(2^{3})^{-2}$

25. $(2 \cdot 3)^{-2}$

26. $3 \cdot 2^{-3}$

27. $\dfrac{3^8 \cdot 3^{-2}}{3^5}$

28. $\left(2^{-1} + 5^{-2}\right)^{-1}$

Evaluate for w = -3, x = -1, y = 2, and z = -2:

29. w^{-2}

30. $(2x)^3$

31. $\dfrac{2y^{-3}}{z}$

32. $\dfrac{6z^w}{2x^w}$

Simplify.

33. $3a^4 \cdot 2a^{-2}$

34. $(5ab)(6a^2 b)$

35. $2x^6(-3x^4)$

36. $(2a)^3$

37. $\dfrac{x^6 \cdot x^{-3}}{x^5 \cdot x^{-4}}$

38. $\dfrac{144a}{80ax}$

39. $\dfrac{20a^2}{110ab}$

40. $\dfrac{44ac}{407ac^2}$

41. $\dfrac{196ab}{154bc}$

42. $\dfrac{169x^3}{26x}$

43. $\dfrac{208x^2y^3}{16x^3y^2}$

44. *Five years ago, Gavin was three times as old as Melissa was. When Melissa was 4 years old, Gavin was four times Melissa's age. How old are Gavin and Melissa now?

Assignment #9.3: Products and Quotients of Algebraic Fractions

Example: Divide $\dfrac{29x^3}{5y^2} \div \dfrac{2x}{10y^5}$

Solution: $\dfrac{29x^3}{5y^2} \div \dfrac{2x}{10y^5} = \dfrac{29x^3}{5y^2} \cdot \dfrac{10y^5}{2x} = \dfrac{290x^3y^5}{10xy^2} = 29x^2y^3$

9.3 Exercises:

Simplify:

1. $(3ab)^2(6a^2b^{-5})$

2. $-2^4x^{-1}y^2(2xy)^{-3}$

3. $3x^{-2}(2x^6 - 5x^2)$

4. $x^{-3}(2x^5 - 4x^3)$

Multiply or divide as appropriate.

5. $\frac{3x^8}{10y^4} \cdot \frac{5y^5}{6x^9}$

6. $\frac{3x^4}{50y^3} \cdot \frac{14y^5}{9x^7}$

7. $\frac{2y^3}{15x^3} \div \frac{-6y^5}{5x^2}$

8. $\frac{2y^3}{45x^{15}} \cdot \frac{-9x^5}{5y^7}$

9. $\frac{48x^3}{25y^{-7}} \div \frac{96y^8}{5x^{-3}}$

10. $\frac{36x^2}{25xy^3} \div \frac{72y^{-1}}{15x^{-1}}$

11. $\frac{7xy^3}{80(xy)^3} \cdot \frac{5yx^3}{28x}$

12. *Reilly is 10 years older than his sister, Annabel. There was a time when he was three times older than she was. In one year he will be twice as old as she will be. How old is Reilly now? Is there unnecessary information given? If so, what is it?

Assignment #9.4: Review of Scientific Notation

Scientific notation is the way of using powers of ten to express very large numbers or very small numbers. For example, instead of writing 0.0000000023, we write 2.3×10^{-9}. The first part, the 2.3, must be between 1 and 10. The rest, the 10^{-9}, is a power of ten.

We can think of 2.3×10^{-9} as the product of two numbers: 2.3 and 10^{-9} (the exponential term). The exponent given to 10 is the number of places the decimal point must be shifted to give the number in its normal decimal form. A *positive* exponent shows that the decimal point is shifted that number of places to the *right*. A *negative* exponent shows that the decimal point is shifted that number of places to the *left*.

Examples of scientific notation.

$10000 = 1 \times 10^4$	$52327 = 5.2327 \times 10^4$
$1000 = 1 \times 10^3$	$8354 = 8.354 \times 10^3$
$100 = 1 \times 10^2$	$472 = 4.72 \times 10^2$
$10 = 1 \times 10^1$	$89 = 8.9 \times 10^1$
$1 = 10^0$	
$1/10 = 0.1 = 1 \times 10^{-1}$	$0.32 = 3.2 \times 10^{-1}$
$1/100 = 0.01 = 1 \times 10^{-2}$	$0.053 = 5.3 \times 10^{-2}$
$1/1000 = 0.001 = 1 \times 10^{-3}$	$0.0078 = 7.8 \times 10^{-3}$
$1/10000 = 0.0001 = 1 \times 10^{-4}$	$0.00044 = 4.4 \times 10^{-4}$

<u>Example</u>: Simplify this expression: $(3 \times 10^5) \times (7 \times 10^{-3})$

Solution: $3 \text{ x } 10^5 \text{ x } 7 \text{ x } 10^{-3} = 3 \text{ x } 7 \text{ x } 10^5 \text{ x } 10^{-3} = 21 \text{ x } 10^2 = 2.1 \text{ x } 10^3$

NOTE: Rule of thumb when converting into scientific notation: If the first part, like 21, gets smaller (2.1) then the exponent on the 10 gets larger and vice-versa. $35 \text{ x } 10^2$ becomes $3.5 \text{ x } 10^3$ whereas $.047 \text{ x } 10^9$ becomes $4.7 \text{ x } 10^7$. Be careful with numbers like $.006 \text{ x } 10^{-5}$ which becomes $6 \text{ x } 10^{-8}$ because as the number .006 became larger (6) the exponent became smaller (-8 is smaller than -5).

Example: Simplify this expression: $(3.2 \text{ x } 10^5) \div (4 \text{ x } 10^{-3})$

Solution: $$\frac{3.2\times10^5}{4\times10^{-3}} = \frac{3.2}{4}\times\frac{10^5}{10^{-3}} = 0.8\times10^5\cdot10^3 = 0.8\times10^8 = 8\times10^7$$

9.4 Exercises:

Simplify the following:

1. $\frac{19x^5}{30y^0} \div \frac{11x^{-5}}{4}$

Write the normal decimal form of each of these scientific notation expressions.

2. 5×10^4

3. 8.7633×10^{11}

4. 3.75×10^{-6}

5. 9.3×10^{13}

6. 2.144×10^{-8}

7. 1.62×10^{-9}

Write the following numbers using scientific notation.

8. 3,000

9. 498,000,000

10. 0.0047

11. 1,200,000,000,000,000

12. 0.00000394

13. 0.00000000000000079

14. $0.0063 \text{ x } 10^{-4}$

15. $47000 \text{ x } 10^{-52}$

16. $6022 \text{ x } 10^{21}$

17. $0.000013 \text{ x } 10^{28}$

Simplify these expressions *without* a calculator, and write the answers in scientific notation.

18. $(7 \times 10^{-6}) \times (8 \times 10^{14})$

19. $(1.5 \times 10^{10}) \div (3 \times 10^5)$

20. $(2 \times 10^4) \times (9 \times 10^{-11})$

21. $\frac{2\times10^6}{5\times10^{-10}}$

22. $\dfrac{1\times10^{-3}}{4\times10^{-1}}$

Simplify these expressions *with* a calculator, and write the answers in scientific notation.

23. $(4.389 \times 10^{26}) \times (2.744 \times 10^{21})$

24. $(3.55 \times 10^{2}) \div (1.13 \times 10^{-2})$

25. *Place eight queens on a chessboard so that no queen is positioned to take any other queen on the next move. (A chessboard is an eight by eight grid and a queen moves in a straight line horizontally, vertically, or diagonally.)

Assignment #9.5: More Scientific Notation

When we add or subtract numbers in scientific notation, they must first be manipulated so the numbers "line up" correctly. For example, to add 354 and 17 you would line up the units' digits on the right (the 4 and the 7) and then do your arithmetic. With scientific notation the "zeroes" must be lined up, which means the exponents on the 10's need to be the same. This often requires one number to be shifted out of scientific notation form.

Example: $3.5 \text{ x } 10^4 + 7.6 \text{ x } 10^3$ (If you first added 3.5 + 7.6 your answer would be incorrect)

Solution: $3.5 \text{ x } 10^4 + 7.6 \text{ x } 10^3 = 3.5 \text{ x } 10^4 + 0.76 \text{ x } 10^4 = 4.26 \text{ x } 10^4$

If you turn these into "regular" numbers you will see how this works:

35000
<u>+7600</u>
42600 which equals $4.26 \text{ x } 10^4$

9.5 Exercises:

Multiply or divide as appropriate.

1. $\dfrac{15x^2}{5y^2} \div \dfrac{3x}{10y^4}$

2. $\dfrac{6x^3}{28y^3} \cdot \dfrac{14y^7}{9x^7}$

Compute and simplify *without* a calculator, and then write the answers in scientific notation. Check your answers with a calculator.

3. $(3 \times 10^{9})(6 \times 10^{7})$

4. $(3 \times 10^{-9})(6 \times 10^{7})$

5. $(3 \times 10^{-9})(6 \times 10^{-7})$

6. $(3 \times 10^{9}) \div (6 \times 10^{7})$

7. $(3 \times 10^{-9}) \div (6 \times 10^{7})$

8. $(3 \times 10^{9}) \div (6 \times 10^{-7})$

9. $(3 \times 10^{-9}) \div (6 \times 10^{-7})$

Use the strategy outlined in the introduction to do the indicated arithmetic.

10. $4 \times 10^{24} + 2 \times 10^{25}$

11. $2.43 \times 10^{19} - 6 \times 10^{17}$

12. $2 \times 10^{108} - 6 \times 10^{109}$

Simplify these expressions *with* a calculator and write the answers in scientific notation.

13. $4.922 \times 10^{18} + 2.938 \times 10^{-17}$

14. $7.85 \times 10^{-8} - 3.44 \times 10^{-6}$

15. $4.389 \times 10^{26} \times 2.744 \times 10^{21}$

16. $3.55 \times 10^{2} \div (1.13 \times 10^{-2})$

17. *The classic tub problem.

a) A cold water faucet fills two thirds of a hot-tub in one hour. How long must the faucet run in order to fill the whole tub?

b) A hot water faucet takes two hours to fill the same tub. What fraction of the tub does it fill in one hour?

c) If both faucets are used together, how long will it take them to fill the tub?

Assignment #9.6: Scientific Notation and Dimensional Analysis

9.6 Exercises:

Simplify these expressions *without* a calculator, and write the answers in scientific notation.

1. $4 \times 10^{34} + 2 \times 10^{35}$

2. $1.01 \times 10^{67} - 8.0 \times 10^{66}$

3. $7 \times 10^{-6} \times 8 \times 10^{14}$

4. $1.5 \times 10^{10} \div (3 \times 10^{5})$

5. $3 \times 10^{20} - 4 \times 10^{21}$

6. $1.2 \times 10^{-4} - 1.2 \times 10^{-3}$

Simplify these expressions *with* a calculator, and write the answers in scientific notation.

7. $4.922 \times 10^{18} + 2.938 \times 10^{17}$

8. $7.85 \times 10^{-8} - 3.44 \times 10^{-6}$

For the following problems, please use dimensional analysis and show your units for each multiplied fraction. Use scientific notation where appropriate.

9. Light from the star Antares, the brightest star in the constellation Scorpio, takes about 520 years to reach Earth. If light travels at 186,000 miles per second, approximately how many miles is it from Antares to Earth? Write your answer in scientific notation.

10. Measles: Unicef is an organization that is working to protect all people in the world from dangerous diseases, such as measles. There is a vaccination against measles that is easy to administer; most of you, if not all of you, have been vaccinated against measles. However, many children in the world are not vaccinated. There are approximately 10 million children born each month on Earth. If twelve doses of measles vaccine cost two dollars and every child needs three doses to be protected

from the measles, how much money would be needed to vaccinate all the children born in one year?

11. A tall, nonfat, peppermint hot chocolate with whipped cream has 45 grams of sugar (no wonder it tastes so good!). Five grams of sugar has 20 Calories (with a big C). If you bought one of these drinks every three days for the entire winter season, how many calories (that's a "lowercase c" calorie) would you have consumed? Winter lasts four months around here and for this problem let's say there are 30 days in each of those months. Please write your answer in both regular and SCIENTIFIC notation and show your dimensional analysis set up.

12. If each of these drinks cost \$3.25, how many dollars would you spend over the course of one winter?

13. How many kids could be vaccinated against measles with the amount of money you calculated in the last problem?

Chapter 10: Square Roots and Pythagoras

Assignment #10.1: Vocabulary of Roots

The word "radical" is from the Latin word *radix*, which means root. Related words in English are radish and eradicate. In mathematics, radicals are roots in the sense of square roots, cube roots, etc. $\sqrt{9}$ is a **radical**, and the 9 is called the **radicand**.

The $\sqrt{\ }$ (the **radical sign**) means to take the **positive square root**. So, $\sqrt{9}$ is 3, not -3, even though (-3)(-3) = 9. If we want the **negative square root** we will write $-\sqrt{9}$. [Note that the square root of x^2 is not always x. (It is not x if x is negative.) We will assume that variables under a radical sign are positive in the problems we do, so that the square root of x^2 is in fact x.]

<u>Classifying the results of taking the square root</u>:
Recall that an **integer** is a positive or negative whole number like 1, -3, 0, or 5. A **rational number** is one that can be expressed as a quotient (or *ratio*) of two integers; examples are 2/5, .56, -7, 0. Most of the numbers you normally think of, including the integers, are rational numbers. But there are other numbers such as π or $\sqrt{2}$ that are **irrational**, i.e. not rational. They are *not* expressible as ratios of integers.

The **real numbers** are all the rationals and irrationals. These are the numbers you are looking at when you look at a number line. But there are numbers that are neither rational nor irrational; they are not able to be located on the number line. These are the **imaginary** or **complex numbers**. An example is $\sqrt{-4}$.

A few generalizations are suggested by these examples.

a) The square root of a perfect square integer (like 64) is an integer (8).

b) The square root of a quotient of perfect squares (like 25/4) is a rational number (5/2).

c) The square root of a positive, non-perfect-square (like 7) is an irrational number (which we denote by $\sqrt{7}$).

d) The square root of a negative number is imaginary. (These will be cruelly shunned in this book.)

10.1 Exercises:

1. Find the square root of 25. Now, using your calculator, find $25^{1/2}$. Coincidence?

2. Fill in the blanks.

The square root of a perfect square integer is an _________.

The square root of a quotient of perfect squares is a _____________.

The square root of a ________ is an irrational number .
The square root of a negative number is _________.

Simplify each expression without a calculator.

3. $\sqrt{64}$ **4.** $\sqrt{25}$ **5.** $\sqrt{1}$

6. $\sqrt{0}$ **7.** $-\sqrt{4}$ **8.** $\sqrt{-4}$

9. $\sqrt{\dfrac{64}{9}}$ **10.** $\sqrt{2.25}$ **11.** $\sqrt{16+9}$

12. $\sqrt{16}+\sqrt{9}$ **13.** $\sqrt{25+144}$ **14.** $\sqrt{25}+\sqrt{144}$

15. $\sqrt{49+576}$ **16.** $\sqrt{49}+\sqrt{576}$ **17.** $\sqrt{25\cdot 4}$

18. $\sqrt{25}\cdot\sqrt{4}$ **19.** $\sqrt{1+24\div 2\cdot 2}$ **20.** $\sqrt{81}\cdot\sqrt{25}$

21. $\sqrt{81\cdot 25}$ **22.** $\dfrac{\sqrt{25}}{\sqrt{36}}$ **23.** $\sqrt{\dfrac{25}{36}}$

24. $\sqrt{\dfrac{49}{100}}$ **25.** $\dfrac{\sqrt{49}}{\sqrt{100}}$ **26.** $\sqrt{5+\sqrt{7+\sqrt{77+\sqrt{16}}}}$

27. *Solve $\sqrt{75-\sqrt{w}} = 8$

Assignment #10.2: Irrational Square Roots; Simplifying Radicals

As you may have noticed, not all numbers are perfect squares. Some numbers, like 2, are not the square of an integer. In other words, they do not have a square root that is an integer. The square root of such a number is a decimal, and as we pointed out above, it is irrational. So it cannot be expressed as a ratio of two integers or even as a repeating or terminating decimal. It is a non-repeating and non-terminating decimal. It goes on forever and does not repeat.

Here are the first 100 digits of $\sqrt{2}$. Examine to see that it is a non-repeating and non-terminating decimal.

1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157 3…

That's too much to write down every time we want to use it. Therefore, for our purposes, we will round irrational square roots when we want to use them. If we need the square root of 2, we will use a calculator, and round the result to 1.414. That will be close enough. After all, 1.414 times 1.414 is 1.999396, which is pretty close to 2.

Sometimes, when we have a number which is not a perfect square, rather than simplifying it, we leave it expressed as a radical so that we do not have to round. In this case, we like to make sure that it is written as simply as possible.

Square roots have the following properties:

$$\sqrt{(a)^2} = \sqrt{a}\sqrt{a} = a \quad \text{(for a>0)}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

We can use these properties to help us simplify square roots. A square root is said to be simplified (also said to be in **simplest radical form)** when all of the following are true:

1. No perfect square factors are under the square root sign.
2. No fractions are under the square root sign.
3. No square roots are in the denominator of a fraction.

<u>Example</u>: Simplify $\sqrt{8}$.

<u>Solution</u>: $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

<u>Example</u>: Simplify $\sqrt{75}$.

<u>Solution</u>: $\sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

10.2 Exercises:

Express each root in simplest radical form. If the number is not an integer, please identify between which two integers it lies. Do not use a calculator.

1. $\sqrt{98}$ **2.** $\sqrt{12}$ **3.** $\sqrt{-121}$

4. $-\sqrt{40}$ **5.** $\sqrt{16}$ **6.** $\sqrt{125}$

7. $\sqrt{175}$ **8.** $\sqrt{0.64}$ **9.** $\sqrt{65}$

Express each root in simplest radical form.

10. $\sqrt{32}$ **11.** $\sqrt{48}$ **12.** $\sqrt{33}$

13. $\sqrt{27}$ **14.** $\sqrt{700}$ **15.** $\sqrt{242}$

16. $-\sqrt{432}$ **17.** $\sqrt{10000}$ **18.** $\sqrt{.0001}$

19. $\sqrt{1.123^2}$ **20.** $-\sqrt{(\sqrt{23})^4}$ **21.** $\sqrt{\sqrt{\sqrt{\sqrt{256}}}}$

22. \$60 is to be divided between Indigo and Andrew so that Indigo's share is 2/3 of Andrew's. How much does Andrew receive?

23. *Sammy Sweettooth stopped by the candy shop to purchase some candy. While she waited to pay for her order, she noticed that the owner of the candy store could weigh out any unit number of ounces of candy under 8 pounds using only seven weights and a balance scale. What were the seven weights?

Assignment #10.3: Computing With Radicals

We can multiply or divide any two real numbers (providing we do not divide by zero.) When the numbers involve radicals, we use the properties of radicals to find the desired product or quotient.

Example: Multiply $\sqrt{8} \cdot \sqrt{6}$

Solution: $\sqrt{8} \cdot \sqrt{6} = \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3} = 2 \cdot 2 \cdot \sqrt{3} = 4\sqrt{3}$

Example: Divide $\sqrt{8} \div \sqrt{2}$

Solution: $\dfrac{\sqrt{8}}{\sqrt{2}} = \dfrac{\sqrt{4}\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}} = 2$

We can also add and subtract any two real numbers. The sum of $\sqrt{7}$ and 2 can be expressed as $2 + \sqrt{7}$. We cannot simplify this further unless we use rational approximations. When we have like radicals, however, we can simplify a sum by using the distributive law and collecting like terms.

Example: Add $3\sqrt{2} + 5\sqrt{2}$

Solution: $3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2} = 8\sqrt{2}$

Example: Subtract $3\sqrt{20} - 7\sqrt{5}$

Solution: $3\sqrt{20} - 7\sqrt{5} = 3\sqrt{4}\sqrt{5} - 7\sqrt{5} = 6\sqrt{5} - 7\sqrt{5} = (6 - 7)\sqrt{5} = -1\sqrt{5} = -\sqrt{5}$

10.3 Exercises:

Perform each of the operations below and simplify. Do not use a calculator.

1. $\sqrt{3} \cdot \sqrt{6}$

2. $\sqrt{3} \cdot \sqrt{2}$

3. $\sqrt{5} \cdot \sqrt{15}$

4. $\sqrt{6} \cdot \sqrt{10} \cdot \sqrt{35}$

5. $\sqrt{8} \cdot \sqrt{125}$

6. $\sqrt{11} \cdot \sqrt{11}$

7. $\sqrt{18} \cdot \sqrt{10}$

8. $\dfrac{\sqrt{10}}{\sqrt{5}}$

9. $\dfrac{\sqrt{28}}{\sqrt{63}}$

10. $\dfrac{\sqrt{12}}{\sqrt{75}}$

11. $\dfrac{\sqrt{10}}{\sqrt{32}}$

12. $\dfrac{\sqrt{96}}{\sqrt{300}}$

13. $3\sqrt{7} + 8\sqrt{7}$

14. $8\sqrt{3} - 12\sqrt{3}$

15. $\sqrt{300} + 2\sqrt{3}$

16. $2\sqrt{10} - 7\sqrt{40}$

17. $\sqrt{24} + \sqrt{54}$

18. $3\sqrt{12} + 2\sqrt{3}$

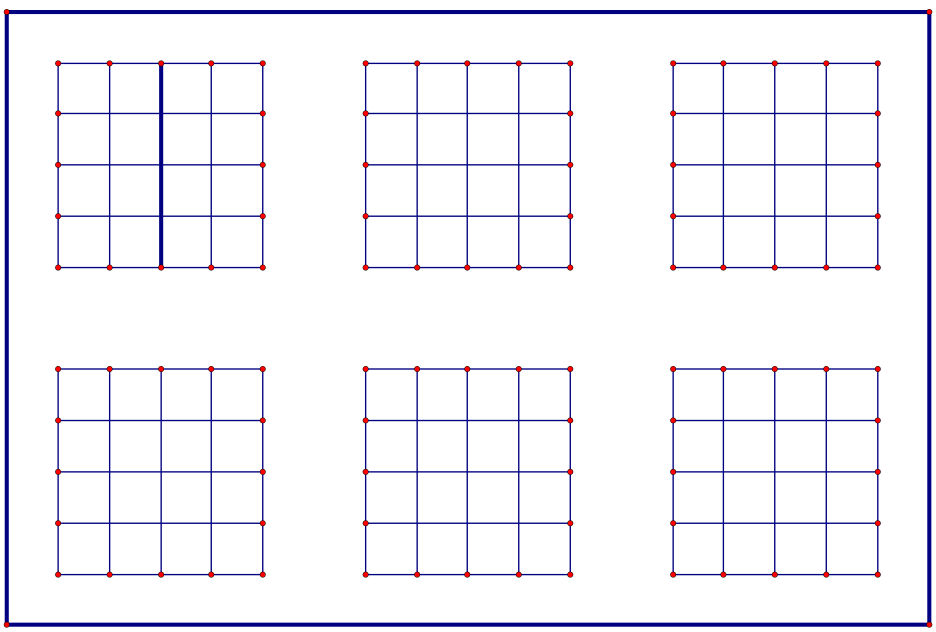
19. $5\sqrt{8} - 5\sqrt{18}$

20. $\sqrt{27} - 2\sqrt{3}$

21. $\sqrt{45} - (\sqrt{16} + \sqrt{20})$

22. $\sqrt{72} - (\sqrt{288} - \sqrt{98})$

23. *One way to cut a 4 by 4 square grid into two ***congruent*** pieces (same size and same shape) is to just cut it down the middle (…as shown in the example). Can you find five other ways to do it? You must stay on the grid lines and different orientations don't count as different solutions.



Assignment # 10.4: Solving with Square Roots

The ability to simplify radicals allows us to solve a new type of equation, those which involve an x^2.

Observing these examples…

$\sqrt{(7)^2} = \sqrt{49} = 7$
$\sqrt{(-2)^2} = \sqrt{4} = 2$
$\sqrt{(8)^2} = \sqrt{64} = 8$
$\sqrt{(0)^2} = \sqrt{0} = 0$

… we can generalize to this property of square roots of perfect squares.

$$\sqrt{x^2} = |x|$$

Thus, we can solve equations with x^2 in the equation.

Example: Solve $x^2 + 11 = 47$

Solution: $x^2 + 11 = 47$
-11 -11
$x^2 = 36$
$\sqrt{x^2} = \sqrt{36}$ (We took the square root of each side.)
$|x| = 6$ (Using the property $\sqrt{x^2} = |x|$ and knowing that $\sqrt{36} = 6$)
x = 6 or -6

Quicker Solution:
$x^2 + 11 = 47$
-11 -11
$x^2 = 36$
x = 6 or -6 (because there are two numbers which we can square to get 36)

Example: Solve $2(b^2 + 5) = 6$

Solution: $2(b^2 + 5) = 6$
$2b^2 + 10 = 6$
- 10 -10
$\frac{2b^2}{2} = \frac{-4}{2}$
$b^2 = -2$
Answer is null set since no number can be squared and give a negative result.

10.4 Exercises:

Simplify.

1. $\sqrt{45} - 2(3 + \sqrt{80})$

2. $3\sqrt{18} - (2\sqrt{32} + 5\sqrt{50})$

3. $\sqrt{2}(\sqrt{12} - 2\sqrt{27}) - 5\sqrt{48}$

Solve. Express your answers in simplest radical form.

4. $b^2 = 100$

5. $x^2 = 72$

6. $x^2 = -25$

7. $y^2 - 40 = 10$

8. $10 = p^2 + 5$

9. $48 = 3y^2$

10. $0 = 5(a^2 - 5)$

11. $\frac{x}{9} = \frac{4}{x}$

12. $-9 = p^2 + 9$

13. $(4r)^2 - 13 = 19$

14. $25 = (r + 1)^2 + 16$

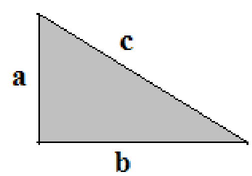
15. $4x^2 + 2 = 50$

16. #From a group of six students, a social committee is to be appointed. The committee must have three members. How many different committees can be formed?

17. *In the year 1849, a person 43 years old squared his age. The result, as you can see, was the same as the year. Sometime in the 20th century another person did the same thing (squared age = the year). In what year was that person born?

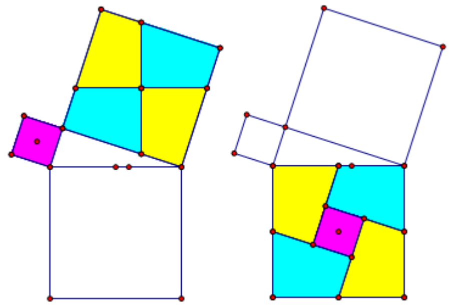
Assignment #10.5: The Pythagorean Theorem

The statement of the Pythagorean Theorem was discovered on a Babylonian tablet from around 1600 B.C. Whether Pythagoras (560-480 B.C.) or someone else from his school was the first to discover a proof of the theorem is not known.

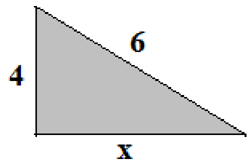


$$a^2 + b^2 = c^2$$

The theorem, namely that the sum of the squares of the legs of a *right* triangle is equal to the square of the hypotenuse, has many famous proofs. One proof can be suggested with a few simple pictures.



Example: Find x



Solution: $x^2 + 4^2 = 6^2$,
$x^2 + 16 = 36$
$x^2 = 20$
$x = \sqrt{20} = 2\sqrt{5} \approx 4.472$ units.

10.5 Exercises:

Solve

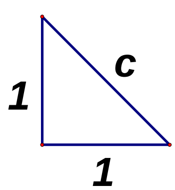
1. $r^2 - 13 = 23$

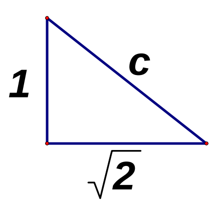
2. $r^2 = r^2 + 3$

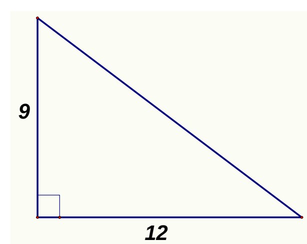
3. $3(p^2 - 2) - 19 = 50$

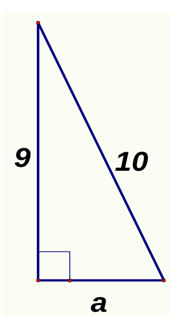
4. $16 = (2 - y)^2 + 16$

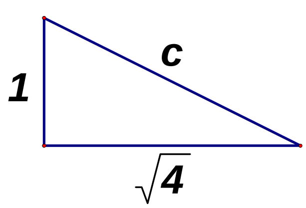
Use the Pythagorean Theorem to find the missing side in each of the right triangles below. Express each answer as a square root (in simplest radical form) and also as a decimal (rounded to the nearest hundredth, if necessary).

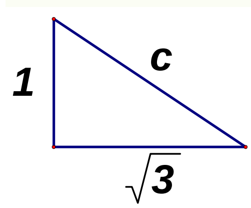
5. 

6. 

7. 

8. 

9. 

10. 

If a and b denote the lengths of the two legs of a right triangle, and c denotes the length of the hypotenuse, find the length of each of the missing sides. Express your answer in simplest radical form.

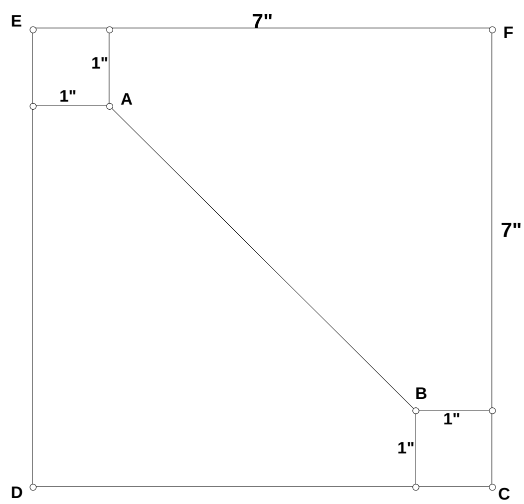
11. a = 3 in, b = 9 in
12. a = 5 cm, c = 9 cm
13. a = 5 mm, c = 13 mm
14. a = 4 m, b = 6 m
15. c = 15 ft, b = 5 ft

16. A straight road climbs 300 vertical feet as you travel a horizontal distance of 3 miles. The road is longer than the horizontal distance since it climbs at an angle. How much longer is the road surface than the horizontal distance? Express your answer as a decimal rounded to the nearest one hundredth. **Start by making a sketch.** The "climb" is the amount the car has lifted from its start point.

17. A 32 foot ladder is leaning against a building with its base 10 feet away from the bottom of the side of the building. The other end is touching the building several feet up in the air. How high above the ground is the top end of the ladder? Round your answer to the nearest tenth. Start by drawing and labeling a sketch.

18. From a group of six students, a social committee is to be appointed. The committee must have at least three members. How many different committees can be formed?

19. *Find the distance from A to B. CDEF is a square with each side = 7 inches. Express your answer in simplest radical form.



Assignment #10.6: Pythagorean Theorem Applications

10.6 Exercises:

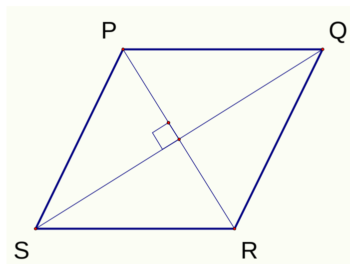
Solve and check. Express all answers in simplest radical form unless directed otherwise.

1. Imagine you are standing at one corner of a rectangular field which measures 70 meters by 155 meters. To get to the opposite corner, you have the choice of walking around the outside of the field or taking a diagonal shortcut straight to the

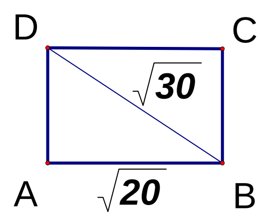
opposite corner. How much distance could you save by taking the diagonal shortcut? Start by drawing and labeling a sketch. Express your answer to the nearest meter.

2. Btown is 52 miles from Kville. Kville is 18 miles due south of C Center. Btown is due east of C Center. How far is Btown from C Center? Express your answer to the nearest mile. Start by drawing and labeling a sketch.

3. If $\overline{PR} = 18$ and $\overline{SQ} = 26$, find the length of the sides of rhombus ***PQRS***. It may help to recall that the diagonals of a rhombus bisect one another. Express your answer in simplest radical form and then as a decimal rounded to the nearest tenth.



4. If $AB = \sqrt{20}$ and $DB = \sqrt{30}$, find the length of $\overline{AD}$. Express your answer in simplest radical form.



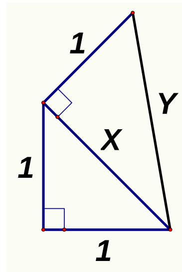
5. What is the perimeter of rectangle ABCD above? Express your answer in simplest radical form.

6. The right fielder for the Mariners catches a pop fly on the right field foul line, in shallow right field, 40 feet beyond first base. How far does he have to throw the ball to get it to the third baseman who is standing on third base? (Hint: Recall that it is 90 feet between the bases on the base path of a baseball diamond.) Express your answer in simplest radical form and rounded to the nearest foot.

7. To the nearest tenth of a centimeter, what is the length of the diagonal of a rectangle with side lengths 5 cm and 2 cm? Start by making a sketch.

8. Trey travels 4 miles east and 7 miles south, then 3 miles east and 1 mile north, and finally, 2 miles west. How far is Trey from his original location? Express your answer to the nearest tenth of a mile.

9. Find the lengths ***X*** and ***Y*** in the figure below. Express your answer in simplest radical form.



Chapter 11: Perimeter, Area, and Volume

Throughout this chapter and beyond, we will use the following formulas for perimeter, area, and volume.

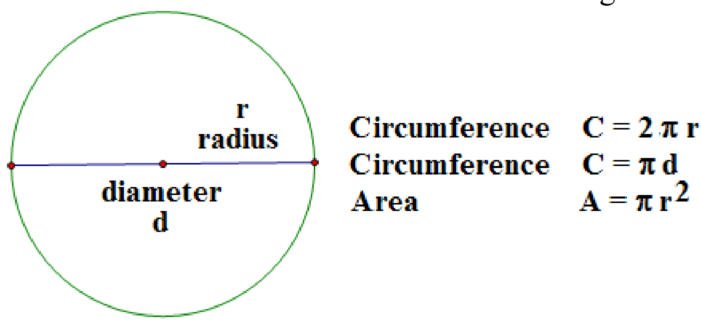
Perimeter formula	
Square	$4 \cdot \text{side}$
Rectangle	$2 \cdot (\text{length} + \text{width})$
Parallelogram	$2 \cdot (\text{side1} + \text{side2})$
Triangle	$\text{side1} + \text{side2} + \text{side3}$
Regular n-polygon	$n \cdot \text{side}$
Trapezoid	$\text{height} \cdot (\text{base1} + \text{base2}) / 2$
Circle	$2 \cdot \pi \cdot \text{radius}$
Area formula	
Square	side^2
Rectangle	$\text{length} \cdot \text{width}$
Parallelogram	$\text{base} \cdot \text{height}$
Triangle	$\text{base} \cdot \text{height} / 2$
Trapezoid	$\text{height} \cdot (\text{base1} + \text{base2}) / 2$
Circle	$\pi \cdot \text{radius}^2$
Cube (surface)	$6 \cdot \text{side}^2$
Sphere (surface)	$4 \cdot \pi \cdot \text{radius}^2$
Cylinder (surface)	$2 \cdot \pi \cdot \text{radius} \cdot \text{height} + 2 \cdot \pi \cdot \text{radius}^2$
Volume formula	
Cube	side^3
Rectangular Prism	$\text{side1} \cdot \text{side2} \cdot \text{side3}$
Sphere	$(4/3) \cdot \pi \cdot \text{radius}^3$
Cylinder	$\pi \cdot \text{radius}^2 \cdot \text{height}$
Cone	$(1/3) \cdot \pi \cdot \text{radius}^2 \cdot \text{height}$
Pyramid	$(1/3) \cdot (\text{base area}) \cdot \text{height}$

Assignment #11.1: Circles: Circumference and Area

A **circle** is a set of points that are all the same distance from a point called the **center**. The distance from the center to any point on the circle is called the **radius**. Twice the radius is the **diameter**, or width, of the circle. The distance around a circle (known as the

perimeter in other contexts) is called the **circumference**. The number of square units covered by the interior of the circle is called the **area** of the circle.

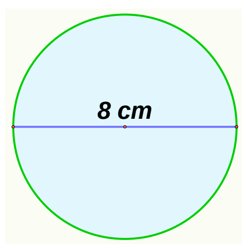
Familiar formulas for the area and circumference of a circle are given below.



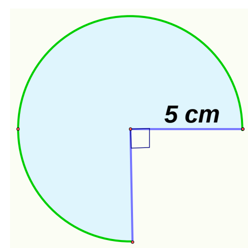
Exercises:

1. Find the circumference (or perimeter) and area of each figure below. Express each answer first in terms of π, then as a decimal rounded to the nearest hundredth.

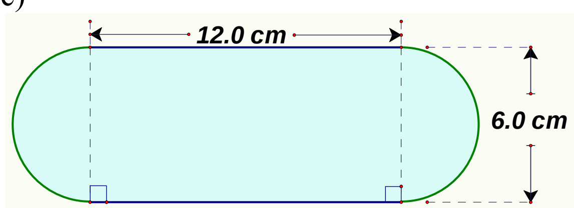
a)



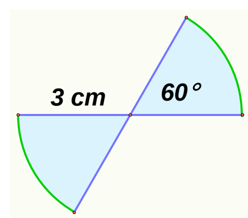
b)



c)



d)



2. What is the combined area of two circles, one of which has a radius of 10 meters and one of which has a diameter of 10 meters?

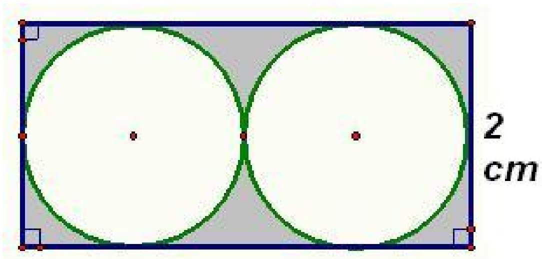
3. Which is closer to π, 3.14 or $\dfrac{22}{7}$? Hint: use your calculator.

4. If the diameter of a circle is $\dfrac{3}{\pi}$ cm, what is the exact circumference of the circle?

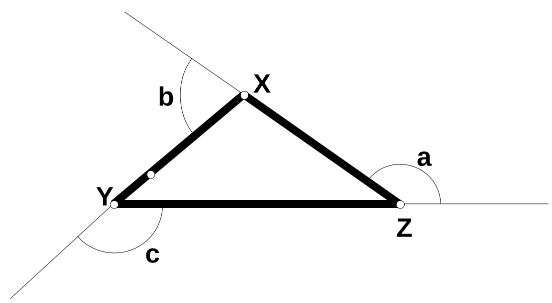
5. If the diameter of a circle is $\dfrac{2.54}{2\pi}$ cm, what is the exact circumference of the circle? Express your answer as a decimal.

6. A wheel has a diameter of 42 inches. About how many feet will it travel in 1 revolution? Round your answer to the nearest integer.

7. A wheel has a diameter of 42 inches. About how many revolutions will it make in traveling 1 mile? Round your answer to the nearest integer.
8. Find the exact area of the shaded region in the figure below.



9. *The thin segments are extensions of the sides of the triangle XYZ. What is the sum: a + b + c?

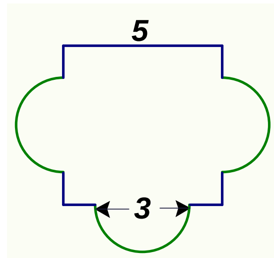


Assignment #11.2: Circle Problems

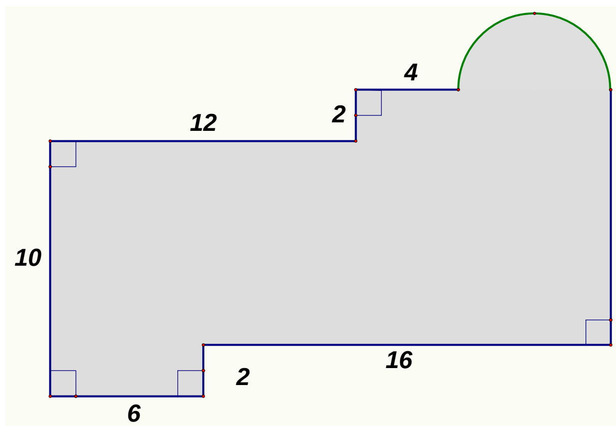
Exercises:

1. If the area of a circle is 144 π square inches, what is its radius?
2. If the area of a circle is 49π square meters, what is its diameter?
3. What is the area of a sector of a circle which has a central angle measuring 40° and a radius of 12 feet? Give an exact answer.
4. The diameter of an American silver dollar is very close to 38 millimeters. In millimeters, what is the circumference of the coin? Express your answer in terms of π and as a decimal to the nearest hundredth.
5. In centimeters, what is the circumference of the coin? Express your answer as a decimal to the nearest hundredth.
6. In square millimeters, what is the area of an American silver dollar? Express your answer in terms of π and as a decimal to the nearest hundredth.
7. In square centimeters, what is the area of an American silver dollar? Express your answer as a decimal to the nearest hundredth.
8. A square with side length 5 has semicircles constructed on three of its sides. The semicircles each have a diameter of 3. Find the perimeter and area of the figure.

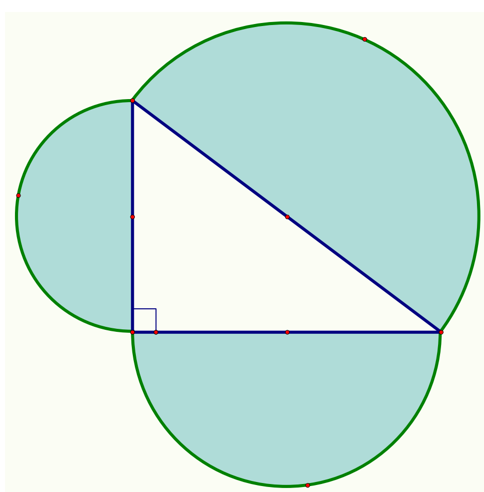
Express each answer first in terms of π, and then as a decimal rounded to the nearest hundredth.



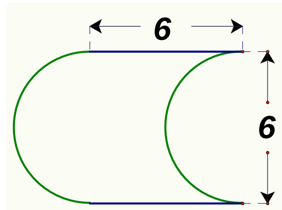
9. Find the perimeter and area of the figure below. Express your answer first in terms of π, and then as a decimal rounded to the nearest hundredth.



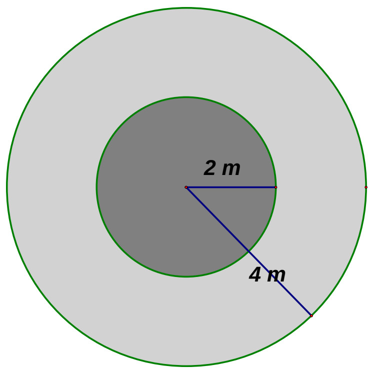
10. Use a centimeter ruler to measure the diameters of the three semi-circles in the figure shown. Use your measurements to calculate the area of each semi-circle. Compare the areas. What do you notice?



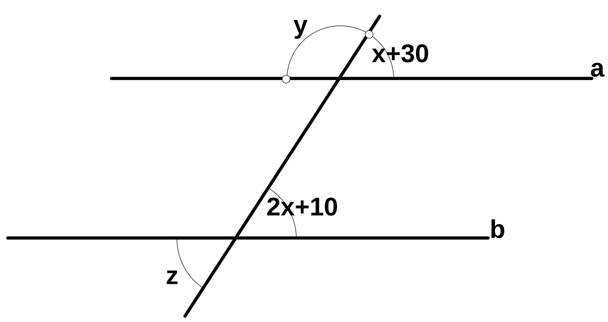
11. Find the perimeter and area of the figure below. Express your answer in terms of π and as a decimal to the nearest hundredth.



12. The *light gray* region in the figure below represents a circular garden path that Emily wants to build around her garden. The garden is represented by the dark gray region. Emily wants to build a fence around the garden and another fence around the outside of the path. What is the total length of fence that Emily must build? Express your answer in terms of π and as a decimal to the nearest hundredth.



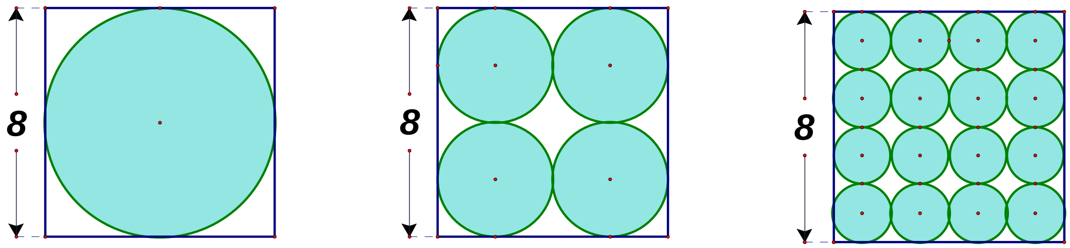
13. *Solve for x, y, and z.



Assignment #11.3: Circle Problems II

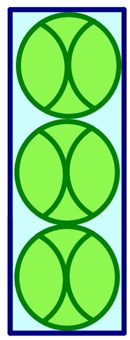
Exercises:

1. Find the total area of the shaded regions in each figure below. Give your answers in terms of π and as decimals to the nearest hundredth.



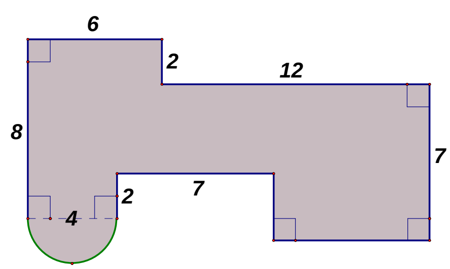
2. What observations can you make based on your results from the problem above?

3. Which is greater, the height of a can tightly holding three tennis balls or the circumference of the same can? Explain your reasoning.



4. A circle with radius 2 inches is cut out of a square piece of tin 4 inches on a side. What is the area of the piece of tin which is left over? Express your answer in terms of π and as a decimal to the nearest hundredth.

5. Find the perimeter and area of the figure below. Express your answer in terms of π and as a decimal to the nearest hundredth.



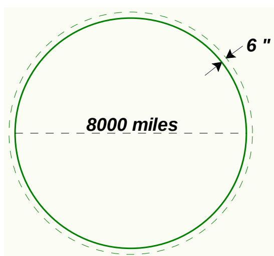
6. Sara has a circular window in her room with an area of 196π square inches. What is the circumference of the window? Express your answer in terms of π and as a decimal to the nearest hundredth.

7. A target is made up of five concentric circles with diameters 2, 4, 6, 8 and 10 inches. What is the area of the shaded part of the target? Express your answer in terms of π and as a decimal to the nearest hundredth.

8. Thi-Nhuy's mountain bike has wheels with diameter 26.5 inches. How many revolutions would her bike wheels make if Thi-Nhuy rode 3.2 miles around Greenlake? Express your answer to the nearest integer.

9. *Assume the earth is a perfect sphere with diameter 8000 miles, and that a metal band is wrapped tightly around the equator.

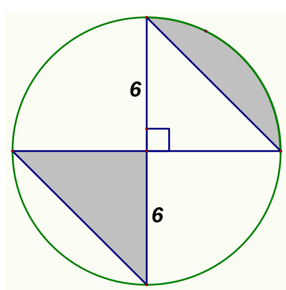
Now suppose the band is lengthened so that circle is now standing out from the earth 6 inches all the way around the planet. How much longer must the band be lengthened in order to stand out 6 inches from the surface?



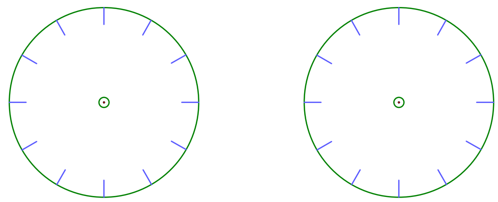
Assignment #11.4: Circle Problems III

Exercises:

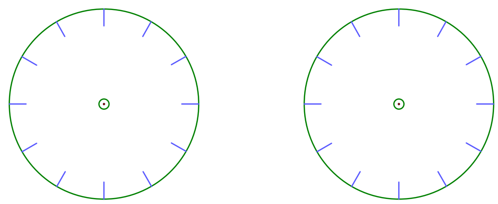
1. Find the area of the shaded region in the figure below. Express your answer in terms of π and as a decimal to the nearest hundredth.



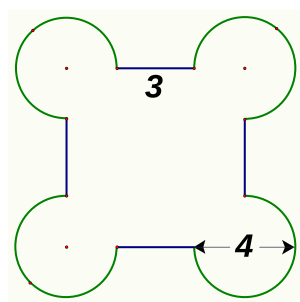
2. What is the obtuse angle formed by the hands of a clock at
a) 4:00? b) 4:05?



3. What is the acute angle formed by the hands of a clock at
a) 4:10? b) 4:20?



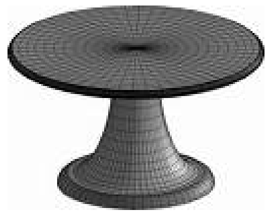
4. Find the perimeter and area of the figure below. Express your answer in terms of π and as a decimal to the nearest hundredth.



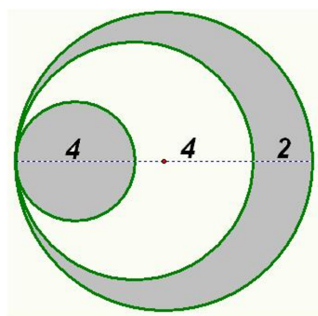
5. If r is the radius of a circle and D is the diameter, which is larger, $2r^2$ or D^2? Explain or show your reasoning.

6. Billy tied his goat to the corner of a rectangular shed. The rope, not including the knots, was 10 meters long and the barn measured 8 meters by 15 meters. Assume the goat could graze on the grass anywhere within 10 meters of where he was tied. What was the total area available for Billy's goat to graze? Express your answer in terms of π and as a decimal to the nearest tenth.

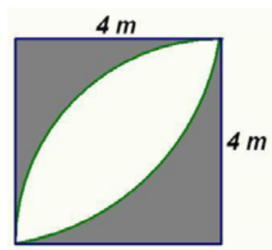
7. A furniture company wants to manufacture a new style table. The top will be circular. The material they will use for the top weighs 2.5 grams per square centimeter of surface area. The base of the table has a limit of 20 kilograms that it can safely support. How many centimeters is the diameter of the largest circular top they can use for their table? Express your answer to the nearest whole unit.



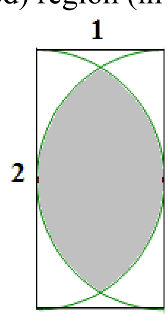
8. Find the area of the shaded region in the figure shown below. (The diameter of the large circle is 10.) Express your answer in terms of π and as a decimal to the nearest tenth.



9. Find the area of the shaded region in the square figure shown below. Express your answer in terms of π and as a decimal to the nearest tenth.

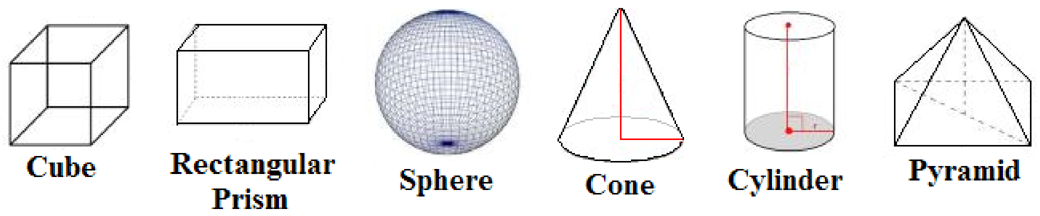


10. **Two overlapping semicircles are inscribed in a 1" by 2" rectangle. Find the area of the overlapping (shaded) region (in terms of π).



Assignment #11.5: Volume

Exercises:



Find the exact volume and surface area of each of the following solids. Be sure to indicate which is which. Start by drawing and labeling a sketch.

1. A cube whose sides are all 6 inches.

2. A rectangular prism whose sides are 1, 5, and 9 inches.

3. A sphere of radius 10 inches.

4. A cylinder of height 12 and radius 3 cm.

5. A cube whose sides are all 8 cm.

6. A rectangular prism whose sides are 3, 5, and 12 cm.

7. A sphere of radius 4 cm.

8. A cylinder of height 9 and radius 5 cm.

9. A triangular prism whose base is a right triangle with legs 3 inches and 4 inches and whose height is 10 inches.

Find the exact volume of each of the following solids. When appropriate, leave your answer in terms of π.

10. A cone of height 12 and radius 3 cm.

11. A pyramid whose base is a 4 inch square and whose height is 12 inches.

12. A cone of height 9 and radius 5 cm.

13. A pyramid whose base is a right triangle with legs 3 inches and 4 inches and whose height is 9 inches.

Answer the following questions by writing and solving an equation.

14. A rectangular prism has a volume of 150 cm^3. If it has a length of 10 cm and a width of 5 cm, what is its height?

15. A cone has a volume of 12π in^3. If it has a height of 9 in, what is the radius of its base?

16. A cylindrical soda can with a volume of 354 cm^3 has a diameter of 6 cm. Calculate the number of centimeters in the height of the can. Express your answer as a decimal to the nearest tenth.

17. A cardboard tube is 5.5 in. long. If an inside diameter of the tube is 1.75 in., how many square inches are in the surface area of the inside of the tube? Express your answer in terms of π.

Chapter 12: Sequences and Series

An ordered list of numbers is called a **sequence**. For example, 5, 10, 15, 20, 25, … is the sequence of positive multiples of 5. The … symbol (known as the **ellipsis**) indicates that the sequence goes on forever in the same manner.

Assignment #12.1: Arithmetic Sequences

An **arithmetic sequence** (put the emphasis on the syllable "met" when pronouncing "arithmetic" in this context) is one in which each term is different from the previous term by the same amount. The amount by which the terms differ is called the **common difference**.

<u>Example</u>: What is the common difference of the following arithmetic sequence?

5, 10, 15, 20, 25, …

<u>Solution</u>: Since each term is 5 more than the previous term, the common difference is 5.

12.1 Exercises:

For each sequence, find the next <u>two</u> terms. Then say if the sequence is an <u>arithmetic</u> sequence.

1. 1, 2, 3, 4, 5, …

2. 1, 4, 9, 16, 25, …

3. 1, 1, 2, 3, 5, 8, 13, …

4. π, 2π, 3π, 4π, 5π, …

5. 2, 4, 8, 16, 32, …

6. 1, 3, 6, 10, 15, 21, …

7. d, g, j, m, …

For each arithmetic sequence below, find the missing terms.

8. 8, 11, 14,__, __, 23, 26,…

9. -5, __, ___, ___, -17, -20, -23,…

10. Find the next number in the following sequence.

5/12 1/3 1/4 1/6 1/12

11. Find the next two numbers in the following sequence.

8, 15, 10, 13, 12, 11, 14, 9, 16, 7, …

12. Suppose that beads are strung on a string in the pattern:

Black, White, Red, Red, Black, White, Red, Red, Black, White, ….

What color is the 127th bead?

13. Determine the units digit of:
a) 2^5 b) 2^{10} c) 2^{15} d) 2^{75}

14. *Find the next three numbers in the following sequence.
3, 13, 1113, 3113, 132113, ____, ____, ____

Assignment #12.2: Arithmetic Sequences II

In general, an arithmetic sequence can be thought of as a first term, which we can call by the letter **a**, and successive terms, each of which is **d** bigger than the last. That is, the series is of the form: a, a + d, a + 2d, a + 3d, a + 4d, etc.

Example: Find the 150th term of the sequence 10, 13, 16, 19, …

Solution: We want to take 10 and add on 149 3s. So the sum is 10 + 149(3) = 457.

Example: Which term is 55 in the sequence 10, 15, 20, ….?

Solution: This sequence starts with 10 so we use that to begin. It goes up by 5 each time. So 10 + 5x = 55, where x is the number of "jumps" of 5 to get to 55. Using algebra to solve the equation, x = 9. This means that nine "jumps" of 5 are needed to get to 55, which makes 55 the 10th term of this sequence.

12.2 Exercises:

1. Find the 20th term of 2, 4, 6, ….

2. Find the 30th term of 3, 6, 9, ….

3. Find the 30th term of 11, 14, 17, ….

4. Find the 40th term of 5, 10, 15, ….

5. Which term is 179 in the sequence 4, 11, 18, ….

6. Which term is 340 in the sequence 20, 28, 36, ….

7. On June 1, Fran does 12 sit-ups. Each subsequent day, she does 2 more sit-ups than she did the day before. How many sit-ups does she do on the 22nd day?

8. On June 1, Fran does 12 sit-ups. Each subsequent day, she does 2 more sit-ups than she did the day before. How many sit-ups does she do on July 15?

9. On June 1, Fran does 12 sit-ups. Each subsequent day, she does 2 more sit-ups than she did the day before. Eventually, there is a day upon which she does 74 sit-ups. What day is this?

10. In an arithmetic sequence, the first term is 13 and the ninth term is 25. What is the tenth term?

11. Find the next four numbers in each of the following sequences.

a) 2, 5, 10, 17, 26, 37, __, __, __, __
b) 2, 3, 5, 9, 17, 33, __, __, __, __

12. Fill in the missing integers and find the 8th term of this arithmetic sequence.
__, __, __, 10, __, __, 16

13. Find the next three numbers in the following sequence.
1, 2, 6, 24, 120, 720, ________, _________, _________

14. Find the next three letters in each of the following sequences.
a) S, S, M, T, W, ____, ____, ____
b) J, F, M, A, M, ____, ____, ____

15. What is the fiftieth digit to the right of the decimal place in the decimal representation of 6/7? (use your calculator to see a pattern)

16. #What is the units digit of 8^{50}?

17. Find the missing integer in the following sequence.
13, 7, 18, 10, 5, __, 9, 1, 12, 6

18. *Find the next three letters in each of the following sequences:
a) T, T, F, S, E, ____, ____, ____

b) M, V, E, M, J, ____, ____, ____

Assignment #12.3: Arithmetic Series

When we add up the numbers of a sequence, the resulting sum is called a **series**. For example, the following is the series obtained by adding up the reciprocals of the even numbers:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+\frac{1}{16}+\ldots$$

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

Example: Find the sum of the arithmetic series 1 + 3 + 5 + 7 + … + 39

Solution: Write the series twice, once forwards and once backwards and add columns.

$$\begin{array}{ccccc} 1\ + & 3\ + & 5\ + & 7\ + \ldots + & 39 \\ \underline{39\ +} & \underline{37\ +} & \underline{35\ +} & \underline{33\ + \ldots +} & \underline{1} \\ 40 & 40 & 40 & 40 & 40 \end{array}$$

To figure out how many columns there are that add to 40, we have to figure out how many terms are in this sequence. It starts with 1 and ends with 39 and goes up by 2 each time. So 39 = 1 + 2x. Solving for x, x = 19 so 39 is the 20th term of this series (1 is the first term and there are 19 jumps of 2 to = 39).

This means we have 20 columns that add to 40 each time, or a total of $20 \cdot 40 = 800$. This sum, 800, is the series added twice since we wrote it in both directions. We must divide our answer by 2 to get the accurate sum of this series. So the total for one series is 400.

12.3 Exercises:

1. Find the 24th term of 5, 10, 15, ….

2. Find the 18th term of 9, 17, 25, ….

3. In an arithmetic sequence, the third term is 10 and the ninth term is 24. What is the eighth term?

4. Find the 527th letter in the sequence MATHMATHMATHMATH…..

For problems 5 through 10, show your work as demonstrated in the example above.

5. Find the sum of 1 + 4 + 7 + 10 + … + 40.

6. Find the sum of 3 + 6 + 9 + 12 + … + 42.

7. Find the sum of 5 + 12 + 19 + 26 + … + 124.

8. Find the sum of the first 20 terms of 2, 4, 6, ….

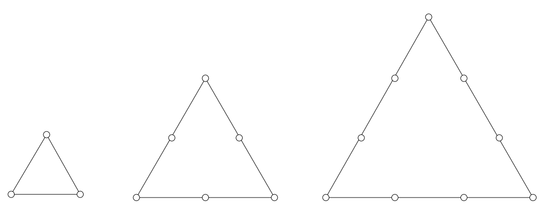
9. Find the sum of the first 30 terms of 3, 6, 9, ….

10. Find the sum of the first 40 terms of 5, 10, 15, ….

11. Reed does 7 pull-ups on September 1. The following day she does 12 pull-ups, the day after that she does 17 pull-ups, and so forth through the entire month of September. How many pull-ups does Reed do in September?

12. Find the sum of the first 20 terms of 100, 98, 96, ….

13. Find the number of line segments in each of the three figures below. (Hint: The second figure contains more than 6 line segments.)



14. *How many line segments will there be in the 5th and 8th figures if this series is continued?

Assignment #12.4: Geometric Sequences

A **geometric sequence** is one in which each term divided by the previous term always gives the same quotient. This quotient is called the **common ratio**.

Example: What is the common ratio of the following geometric sequence?

2, 6, 18, 54, …

Solution: Since each term is 3 times the previous term, the common ratio is 3.

12.4 Exercises:

1. Find the sum of 5 + 11 + 17 + … + 113.

2. Find the sum of the first 25 terms of 3, 5, 7, ….

3. There are 35 seats in the front row of Page Auditorium. After that, each row contains two more seats than the row in front of it. If there are 20 rows of seats in the hall, what is the seating capacity of the auditorium?

For each sequence, find the next two terms. Then say if the sequence is arithmetic, geometric, or neither.

4. 10, 20, 30, 40, 50, …

5. $\pi, \pi^2, \pi^3, \pi^4, \pi^5, \ldots$

6. 1, -2, 4, -8, …

7. 1, 4, 9, 16, 25, …

8. 3, 15, 75, …

9. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots$

10. 11, 11, 22, 33, 55, 88, …

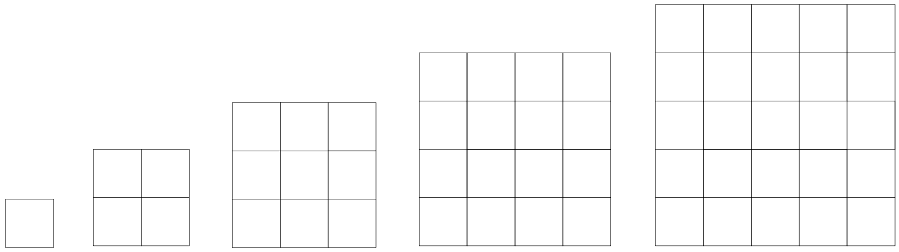
11. $6, -3, \frac{3}{2}, \frac{-3}{4}, \ldots$

12. 6, 12, 24, 48, 96, …

13. 6, 10, 15, 21, 28, …

14. Find the number of squares in each of the five figures below.
(Hint: The third figure contains more than 10 squares.)

15. *How many squares would be in the eleventh figure?



16. *How many rectangles would be in the tenth figure?

Assignment #12.5: Geometric Sequences and Series

In general, a geometric sequence can be thought of as a first term, which we can call by the letter **a**, and successive terms, each of which is **r** times the last.

The sequence can be thought of as: a, ar, ar^2, ar^3, ar^4, ar^5, …

Example: Find the 6^{th} term of the sequence 3, 6, 12, …

Solution: We want to take 3 and multiply it by 2 five times. So the result is $3(2^5)$ which is 96.

A **geometric series** is the sum of the terms of a geometric sequence.

Example: Find the sum of the geometric series:
3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768 + 1536 + 3072 + 6144

Solution: Call the sum S.
S = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768 + 1536 + 3072 + 6144

Now multiply both sides of this equation by 2 (the common ratio).
2S = 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768 + 1536 + 3072 + 6144 + 12288

Now subtract the first equation from the second. Lots of terms go away!
S = 12288 – 3, which is 12,285.

This is a technique of awesome power!

12.5 Exercises:

For each series, find the sum. Do not just enter all the terms into your calculator. Use the techniques we have learned.

1. Find the sum of the first 20 terms of 4, 6, 8, ….
2. 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768
3. 4 + 12 + 36 + 108 + 324 + 972 + 2916 + 8748 + 26244
4. 5 + 20 + 80 + 320 + 1280 + 5120 + 20480 + 81920 + 327680 + 1310720
5. 6 + 30 + 150 + 750 + 3750 + 18750 + 93750 + 468750 + 2343750 + 11718750 + 58593750
6. Find the sum of the first 8 terms of 3, 6, 12, ….
7. Find the sum of the first 7 terms of 1, 5, 25, ….
8. Find the sum of the first 16 terms of 2, 4, 6, ….

9. Find the sum of the first 12 terms of 1, 2, 4, ….

10. Find the sum of the first 14 terms of 3, 6, 9, ….

11. *For what value of **a** does
$1 + 2 + 3 + 4 + \ldots\ldots + \mathbf{a} = 4656$?
(*Hint: write this sum down twice and add the results.)*

Assignment #12.6: Infinite Series

Strangely, a geometric series can go on forever and still add up to a finite sum. This is highly counter-intuitive! How could we add up numbers forever and have the result converge to a single number? An example will help.

Example: Add up the infinite geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}\ldots$

Solution: First, let's look at the partial sums:

$$1+\frac{1}{2}=\frac{3}{2}=1\tfrac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}=1\tfrac{3}{4}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}=1\tfrac{7}{8}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{31}{16}=1\tfrac{15}{16}$$

We can see that the sum is getting closer and closer to 2, with each step closer being half as big as the previous step. So the further we continue the series, the closer the sum gets to 2. We can get as close as we want to 2 by simply taking more terms in the series. *That is what we shall mean* when we say that the sum of the infinite series is 2.

Now let's try a more formal version of getting the sum. Call it S, as usual:

$$S = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots$$

Now multiply both sides of this equation by ½ (the common ratio).

$$\tfrac{1}{2}S = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots$$

Now subtract the second equation from the first. Lots of terms go away!
$\tfrac{1}{2}S = 1.$
Therefore, if half of S is 1, then S must be 2. Amazing!

12.6 Exercises:

For each series, find the sum. Do not just enter a lot of terms into your calculator. Use the techniques we have learned.

1. Find the sum of the first 20 terms of 3 + 10 + 17 + ….

2. Find 5 + 10 + 20 + … + 5120.

3. Find $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}...$

4. Find the sum of the first 20 terms of 3, 6, 12, ….

5. Find the sum of the first 20 terms of 4, 8, 12, ….

6. Find $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}...$

7. Find the sum of the first 7 terms of 4, 8, 16, ….

8. Find $1+\frac{1}{6}+\frac{1}{36}+\frac{1}{216}...$

9. Find the sum of the first 9 terms of 1, 6, 36, ….

10. Find the sum of the first 30 terms of 0.5, 1.5, 2.5, ….

11. Find the next three terms in the sequence 1, 2, 3½, 5½, 8, 11, . . .

12. *What do you observe about the infinite sum 1 + -1 + 1 + -1 + 1 + -1 . . . . ?

13. *According to an ancient belief, when a friend visits a sick person, 1/60 of his or her illness is taken away. How many friends need to visit in order to take away at least 99% of a person's illness? (*Hint: Think about geometric series. After one person visits, 59/60 of the illness remains, so the next visitor takes away 1/60 of that amount, etc.)*